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Modal Characteristics of Composite Beams with Single Delamination- A Simple Analytical Technique

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ABSTRACT

In the present research, the effects of delamination size and location on vibration characteristics of laminated composite beams are investigated both analytically and numerically. In the analytical method, the delaminated beam is modeled as four interconnected Euler–Bernoulli beams and the constrained and free mode models are both simulated. The differential stretching and the bending-extension coupling are considered in the formulation. Analytical expressions for displacement functions are presented in a simple form based on the basic standard trigonometric and hyperbolic functions. This new technique considerably simplifies the calculations regardless of the number of delaminations. In finite element method, delaminated composite beams are modeled in commercial finite element software, ABAQUS, and natural frequencies and mode shapes are extracted from the modal analysis. Analytical results are compared with finite element ones and available experiments in the literature. Finally, the generated database for different sizes and locations of delamination can be used to detect the existence of delamination in laminated composite beams and thento specify the size and location of delamination.

1. Introduction

Advanced composite materials are increasingly used in structural designs of aircraft, helicopters, spacecraft, automobiles, marine and submarine vehicles because of the desirable properties such as high strength and stiffness, lightweight, fatigue resistance, and damage tolerance, etc. However, composites are very sensitive to the anomalies induced during their fabrication or service life. One of the commonly encountered types of defects or damage in laminated composite structures is delamination. Delaminations may originate during fabrication or may be service-induced, such as by impact or fatigue loading. Delaminations not only affect the strength and integrity of the structure but also cause the reduction in the stiffness, thus affecting its vibration and stability characteristics. Reflections of these effects in dynamic response are the alteration of natural frequencies and damping ratios. As a result, considerable analytical, numerical and experimental efforts have been expended to capture these phenomena.

One of the earliest models for vibration analysis of composite beams including delaminations was proposed by Ramkumar et al. [1]. They modeled a beam with one through-thickness delamination simply using four Timoshenko beams connected at delamination edges. Natural frequencies and mode shapes were solved by a boundary eigenvalue problem. Using this model, the predicted natural frequencies were consistently lower than the results reported in

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experimental measurements. Authors attributed this discrepancy to the effect of contact with the delaminated "free" surfaces during vibrations. They suggest that the inclusion of the contact effect may improve the analytical prediction.

To study the effect of a through-thickness delamination on the free vibration of an isotropic beam, Wang et al. [2] present an analytical model using four Euler-Bernoulli beams that are joined together. They assum that the delaminated layers deform "freely" without touching each other ("free mode" model) and will have different transverse deformations.

Later, Mujumdar and Suryanarayan [3] presented two models namely the free mode model and the constrained mode model for the flexural vibrations of isotropic beams with delamination at the mid-plane as well as at the off-mid-plane locations. Experimental results have also been presented for various cases of delaminations in the beams. This paper concludes that the constrained mode model in which the transverse displacement and the normal stress of the upper and lower layer at the delaminated interface have been constrained to be the same, gives results in good agreement with the experimental results. The free mode model (similar to the one developed by Wang [2]) underestimates the frequencies, particularly at the higher modes.

The "constrained model" fails to explain the delamination opening modes found in experiments [4]. In these experiments conducted by Shen and Grady, the opening modes were even found in the first bending mode of the beam for some delamination cases. However, their finite element formulations (Model A and B) were essentially followed by the "constrained model" by Mujumdar and Suryanarayan [3] and the "free model" by Wang et al. [2]. In this paper the "Model A" is corresponding to the "constrained model" and the "Model B" is corresponding to the "free model". The discrepancy between the results predicted by the two models is significant even in cases where mode shapes do not show any opening in the delamination region. Furthermore, in some cases, opening delamination modes are shown clearly in their experiment, while the "constrained model" frequency prediction has a better match with the corresponding experimental results for these modes, even though the delamination cannot open using the "constrained model".

An analytical model based on the Timoshenko beam theory is presented by Hu and Hwu [5] for the free vibrations of delaminated sandwich beams. The natural frequencies and the mode shapes of the delaminated composite sandwich beams are presented in this paper. Lee [6] presents a displacement-based layer-wise finite element model for the analysis of free vibration of delaminated beam. In which the effects of the fiber angle, location, size and number of delamination are investigated numerically. A review paper on the vibration-based model-dependent damage (delamination) identification and health monitoring for composite structures is presented by Zou et al. [7]. This paper deals with various models proposed for the free vibrations of delaminated beams.

In the analytical research done by Luo and Hanagud [8], a piecewise-linear spring model is used to simulate the behavior between delaminated surfaces. Shear and rotary inertia effects, as well as bending-extension coupling, are included in the governing equations. Frequencies and mode shapes are solved through a boundary eigenvalue problem. The proposed model includes the "free model" and the "constrained model" as special cases. The nonlinear response simulated by this model shows good agreement with the experiment results.

Karmakar et al. [9] studied the effect of delamination on free vibration characteristics of graphite-epoxy composite pre-twisted shallow shells of various stacking sequences considering length of delamination as a parameter. An exhaustive review on the vibration of delaminated composites has been presented by Della et al. [10]. The paper deals with various analytical models and numerical analysis for the free vibration of composite laminates. Della et al. [11] develop analytical solutions to study the free vibrations of multiple delaminated beams under axial compressive loadings. The Euler–Bernoulli beam theory and free mode and constrained mode assumptions in delamination buckling and vibration are used in the analysis. Ramtekkar [12] presented free vibration analysis of laminated beams with delamination using mixed finite element model. Analytical solutions for beams with multiple delaminations were presented by some researchers. Shu and Della [13, 14] and Della and Shu [15] used the "free mode" and "constrained mode" assumptions to study a composite beam with various multiple delamination configurations. Their study emphasized the influence of a second delamination on the first and second natural frequencies and the corresponding mode shapes of a delaminated beam. Shariati Nia et al. [16] presented an analytical method for calculating natural frequencies of a delaminated composite beam from both free and constrained mode frequencies. An improved combined natural frequency is proposed in this new formulation based on the breathing of delamination.

In this paper, at first, the delaminated beam is modeled as four interconnected Euler-Bernoulli beams. The free mode and constrained mode models are considered. Furthermore, the expressions for displacement functions are presented in a more convenient form based on concerning basic standard trigonometric and hyperbolic functions. The proposed technique considerably simplifies the calculation, and regardless of the number of delaminations, the problem can be solved by a 2×2 system of equations. Also, delaminated composite beams with different sizes and locations of delaminations are modeled in commercial finite element software, ABAQUS, and natural frequencies are extracted from the modal analysis.

2. Analytical Modelling of a Delaminated Composite Beam

Fig 1.(a) shows a cantilever beam with a single delamination at an arbitrary location and various lengths. L, h and a are beam length, beam thickness and delamination length, respectively.

The beam is separated along the interface by a delamination with length a located at the centre of the beam. The beam can be subdivided into three spanwise regions, a delamination region and two integral regions. The delamination region is comprised of two segments (delaminated layers), beam 2 and beam 3, which are joined at their ends to the integral segments, beam 1 and beam 4. Each of the four beams is treated as beams with different boundary conditions.

2.1. Free Mode Model

In "free mode" model, it is assumed that the delaminated layers deform "freely" without touching each other and have different transverse deformations. The governing equations for the free vibration of a delaminated beam using the Euler–Bernoulli beam theory are as the following:

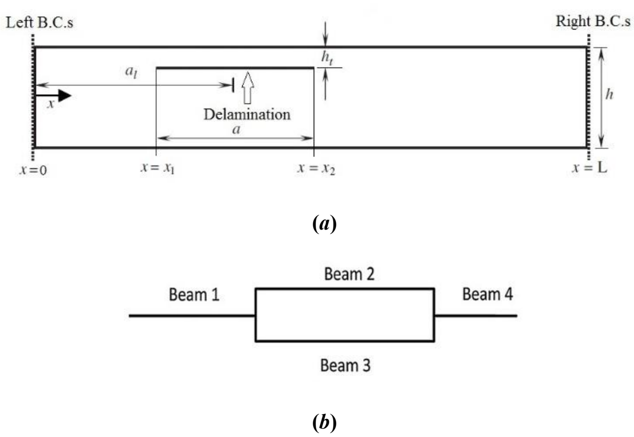


Figure 1. a) Geometry of the delaminated beam, b) Model of delaminated beam with four interconnected beams

$$D_i\frac{\partial^4 w_i}{\partial x^4}+m_i\frac{\partial^2 w_i}{\partial t^2}=0 \quad i=1,2,3,4 \tag{1}$$

where D_i is the equivalent bending stiffness of the ith beam, m_i is the mass per unit length and is equal to $\rho_i A_i$, ρ_i is the mass density and A_i is the cross-sectional area of the beam. The bending stiffness for homogeneous and isotropic beams is given by $D_i=EI_i$, where E denotes the Young's modulus and I is the moment of inertia. The mechanical properties of the composite beams are determined using the classical lamination theory (CLT) as follows:

$$D_i = {D_{11}}^{(i)}-\frac{\left({B_{11}}^{(i)}\right)^2}{{A_{11}}^{(i)}} \tag{2}$$

where ${A_{11}}^{(i)}$ is the extensional stiffness, ${B_{11}}^{(i)}$ is the bending-extension coupling stiffness, and ${D_{11}}^{(i)}$ is the bending stiffness of the ith beam [13].

For free vibration, the solution of Eq. (1) is given as follows:

$$w_i(x_i,t)=W_i(x_i)\,\mathrm{e}^{\mathrm{i}\omega t} \tag{3}$$

where ω is the natural frequency and W_i is the mode shape. Substituting Eq. (3) for Eq. (1) and eliminating the trivial solution $\sin(\omega t)$=0, one can obtain the general solutions of the differential equation in Eq. (1) as follows:

$$\begin{aligned}W_i(x)=C_i\cos(\lambda_i x)+S_i\sin(\lambda_i x)\\+CH_i\cosh(\lambda_i x)\\+SH_i\sinh(\lambda_i x)\end{aligned} \tag{4}$$

where,

$${\lambda_i}^4=\frac{m_i\omega^2}{D_i} \tag{5}$$

and λ_i are the non-dimensional frequencies. The lowest eigenvalue λ is the nondimensional primary frequency of the beam. The 16 unknown coefficients C_i, S_i, CH_i and SH_i are determined by four boundary conditions and twelve continuity conditions.

For cantilever beam B.C.'s at fixed end are as follows:

$$W_1=0\ ,\ W_1'=0 \tag{6-1}$$

and at free end (x=L)

$$W_4''=0\ ,W_4'''=0 \tag{6-2}$$

Continuity conditions at x=x_1

$$W_1 = W_2\,, W_1 = W_3 \quad (7\text{-}1)$$

$$W_1' = W_2', W_1' = W_3' \quad (7\text{-}2)$$

$$D_1 W_1''' = D_2 W_2''' + D_3 W_3''' \quad (7\text{-}3)$$

$$D_1 W_1'' + \beta[W_1'(x_1) - W_1'(x_2)] = D_2 W_2'' + D_3 W_3'' \quad (7\text{-}4)$$

(x=0)

where,

$$\beta = \frac{h^2}{4a}\left[\frac{{A_{11}}^{(2)}{A_{11}}^{(3)}}{{A_{11}}^{(2)} + {A_{11}}^{(3)}}\right] \quad (8)$$

The second term on the left side of Eqs. (7-4) represents the contribution to the bending moment from the differential stretching between beams 2 and 3 and contributes to the bending stiffness of the beam [3, 13-15]. Similarly, the continuity conditions at the end tip of delamination (x=x_2) are derived. The boundary conditions and continuity conditions provide 16 homogeneous equations for 16 unknown coefficients. A non-trivial solution for the coefficients exists only when the determinant of the coefficient matrix vanishes. The frequencies and mode shapes can be obtained as eigenvalues and eigenvectors, respectively.

2.2. Constrained Mode Model

The "constrained mode" model is simplified by the assumption that the delaminated layers are in touch along their whole length all the time, but are allowed to slide over each other. Therefore, the delaminated layers have the same transverse deformations (w_3=w_2). This is reasonable since if there is no opening in the delamination region, the "free model" and "constrained model" are essentially the same. In this model the governing equations are as the following:

$$D_i \frac{\partial^4 w_i}{\partial x^4} + m_i \frac{\partial^2 w_i}{\partial t^2} = 0 \quad i = 1,4 \quad (9\text{-}1)$$

For beams 2 and 3

$$D_{23} \frac{\partial^4 w_2}{\partial x^4} + m_{23} \frac{\partial^2 w_2}{\partial t^2} = 0 \quad (9\text{-}2)$$

where,

$$D_{23}=D_2 + D_3 \quad (10\text{-}1)$$

$$m_{23}=m_2 + m_3 \quad (10\text{-}2)$$

The generalized solutions for the constrained model are identical in form to the free model. The unknown coefficients C_i, S_i, CH_i and SH_i, however, are reduced to twelve coefficients which can be determined by four boundary conditions and eight continuity conditions.

B.C.'s for cantilever beam at fixed end (x=0) and free end (x=L) are the same as Eqs. (6-1) and (6-2), but four continuity conditions at x=x_1 are as follows:

$$W_1 = W_2 \quad (11\text{-}1)$$

$$W_1' = W_2' \quad (11\text{-}2)$$

$$D_1 W_1''' = D_{23}\, W_2''' \quad (11\text{-}3)$$

$$D_1 W_1'' + \beta[W_1'(x_1) - W_1'(x_2)] = D_{23}\, W_2'' \quad (11\text{-}4)$$

Similarly, four continuity conditions at the end tip of delamination (x=x_2) can be derived. These boundary and continuity conditions provide 16 homogeneous equations for 16 unknown coefficients.

According to these procedures, the characteristic equation of a beam with one delamination relies on the solution of a determinant of order 16 for free mode model and 12 for constrained mode model. Furthermore, this determinant order increases by varying the number of delamination and converting single delamination to multiple delaminations. This method is computationally expensive to analyze the presence of multiple delaminations in beams.

2.3. Basic Functions

The expressions for the displacements of beams as general solution of the differential equations in Eq. (9) can be written in a simple form based on the basic standard trigonometric and hyperbolic functions and in terms of just four factors: the displacement W_0, slop θ_0, bending moment M_0, and shear force V_0 at x=0.

$$w(x) = W_0 g_1(x) + \theta_0 g_2(x) + M_0 g_3(x) + V_0 g_4(x) \quad (12)$$

where the functions of $g_i(x)$ are expressed as follows:

$$g_1(x) = \frac{1}{2}[\cos(\lambda x) + \cosh(\lambda x)] \quad (13\text{-}1)$$

$$g_2(x) = \frac{1}{2\lambda}[\sin(\lambda x) + \sinh(\lambda x)] \quad (13\text{-}2)$$

$$g_3(x) = \frac{1}{2\lambda^2}[\cos h(\lambda x) - \cos(\lambda x)] \quad (13\text{-}3)$$

$$g_4(x) = \frac{1}{2\lambda^3}[\sinh(\lambda x) - \sin(\lambda x)] \quad (13\text{-}4)$$

2.3.1. Constrained Mode Model

According to basic functions expressed in Eq. (13) and the boundary conditions at fixed end of beam 1, transverse deformations of beam 1 can be written as follows:

$$w_1(x) = M_0 g_3(x) + V_0 g_4(x) \tag{14}$$

Based on two continuity conditions at $x=x_1$ for displacement and slope, and according to Eqs. (11-1) and (11-2), deflection of beam 2 is as follows:

$$w_2(x) = w_1(x) + \bar{M}_0 g_3(x - x_2) + \bar{V}_0 g_4(x - x_2) \tag{15}$$

These unknown coefficients ($\bar{\boldsymbol{M}}_{\boldsymbol{0}}, \bar{\boldsymbol{V}}_{\boldsymbol{0}}$) can be obtained using two other continuity conditions at $x=x_1$ for bending moment and shear force, which are expressed in Eq. (11-3) and (11-4) as follows:

$$\bar{M}_0 = \left(\frac{D_1}{D_{23}} - 1\right) W_1''(x_1) + \frac{\beta}{D_{23}}[W_1'(x_1) - W_4'(x_2)] \tag{16-1}$$

$$\bar{V}_0 = \left(\frac{D_1}{D_{23}} - 1\right) W_1'''(x_1) \tag{16-2}$$

Similarly, based on two continuity conditions at $x=x_2$ for displacement and slope, deflection of beam 4 is as the following:

$$w_4(x) = w_2(x) + \bar{\bar{M}}_0 g_3(x - x_2) + \bar{\bar{V}}_0 g_4(x - x_2) \tag{17}$$

Also, these unknown coefficients can be obtained from two other continuity conditions at $x=x_2$ for bending moment and shear force as follows:

$$\bar{M}_0 = \left(\frac{D_{23}}{D_4} - 1\right) W_2''(x_2) - \frac{\beta}{D_4}[W_1'(x_1) - W_4'(x_2)] \tag{18-1}$$

$$\bar{V}_0 = \left(\frac{D_{23}}{D_4} - 1\right) W_2'''(x_2) \tag{18-2}$$

Finally, the other two unknown coefficients (M_0, V_0), are obtained by using two equations from boundary conditions at free end of beam 4. Natural frequencies and mode shapes can be obtained by solving this simple system of equations.

The general solution proposed in this method satisfies the compatibility conditions at the delamination tips, and two of the factors are obtained directly from the type of support at x=0. The other two factors are obtained by the system of two equations from the boundary conditions at $x=L$. The technique used considerably simplifies the calculation, and regardless of the number of delaminations, the problem can be solved by a 2×2 system of equations.

Table 1. Mechanical properties of the laminated beam

E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	υ_{12}	ρ (kg/m^3)
134.49	10.34	5.00	0.33	1500

2.4. Numerical Investigations

Numerical investigations are undertaken for the free vibrations of cross-ply laminated beams with embedded internal delaminations of different sizes and at several different locations. For this purpose, an eight-layer symmetric cross-ply laminated $[0/90]_{2s}$ beam, under the clamped-free supports is considered. Thickness direction delamination locations are defined as mentioned in work of Shen and Grady [4]. "Interface 1" implies the mid-plane delamination, while "Interface 4" implies the skin ply delamination and so on. Lengthwise delamination locations are at the middle of beams. Fig. 2 shows interfaces and dimensions for a cross-ply delaminated beam.

3. Finite Element Modelling

The delamination is introduced as a debonding of adjoining plies in the laminated composite beam. The geometry of the delaminated beam is shown in Fig 1. The finite element modelling for simulating the delaminated beams and extracting the natural frequencies is conducted using the commercial finite element software ABAQUS. The composite beam is modeled as a cantilever with length (L) 127 mm, width (b) 12.7 mm and total thickness (h) 1.016 mm with delamination of length of 25.4, 50.8, 76.2, and 101.6 mm along one of the four ply interfaces shown in Fig 2. Nominal thickness of each ply is 0.127 mm and delamination dimensionless length (a/L) is 0.2, 0.4, 0.6, and 0.8. Material properties of unidirectional composite ply are presented in Table 1.

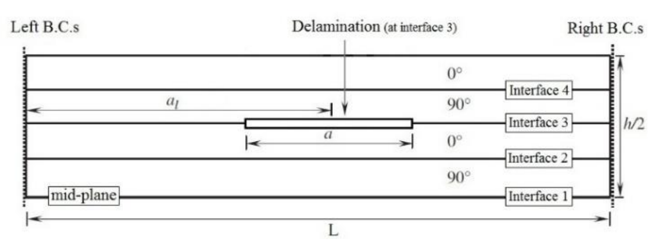


Figure 2. Geometry of the eight layer symmetric composite laminated beam $[0/90]_{2s}$ (Various interfaces for delamination)

A shell element formulation is adopted since the beam is comparatively thin (h/b is less than 10 and h/L is less than 100). The composite laminates are modeled using a double layer of shells, with reduced integration and three integration points over the thickness (ABAQUS element type S4R, Simpson's rule for shell section integration), and an orthotropic material definition is used. Modelling a delamination generally involves nonlinearities, due to the opening and closing of the delamination during cyclic deformation of the structure and friction between the two faces of the delamination. This requires explicit solvers and time-domain simulations.

At "free mode" model, there is no interaction defined between the surfaces of the delamination. Consequently, the elements on either side of the delamination deform freely and can separate and penetrate each other, effectively, causing a lower stiffness and absence of damping induced by contact during cyclic closing of the delamination. At "constrained mode" model, standard surface-to-surface contact interactions with normal and tangential behavior are introduced between the surfaces of the two sub-laminates, which allow neither penetration nor separation between the sub-laminate structures. Therefore, the delaminated layers have the same transverse deformations. Based on the model proposed and material properties, a total of 33 cases including an intact beam and 32 delaminated beams with different delamination lengths and different delamination interfaces are calculated. Natural frequencies are compared with the experimental results provided in Ref. [4].

4. Results and Discussion

The fundamental frequencies of a delaminated composite beam with various locations and sizes of delamination are presented in Tables 2-5. The experimental and analytical results related to model A (constrained model) and model B (free model) are provided from Ref. [4]. It is observed that the fundamental frequencies obtained from analytical and finite element method, closely match the experimental results of Ref. [4] for most of the cases.

Table 2. Fundamental natural frequency (Hz), Interface 1

Shen and Grady [4]				Present Study			
$\frac{a}{L}$	Exp	Analytical		FEM		Analytical	
		Const	Free	Const	Free	Const[a]	Const[b]
0.0	79.83	82.04	82.04	82.12	82.12	82.21	82.21
0.2	78.17	80.13	67.36	80.17	68.67	80.63	80.41
0.4	75.37	75.29	56.48	75.60	58.09	76.10	75.27
0.6	67.96	66.94	47.90	66.64	48.97	68.04	66.4
0.8	57.54	57.24	40.59	56.18	41.16	58.47	56.23

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Table 3. Fundamental natural frequency (Hz), Interface 2

Shen and Grady [4]				Present Study			
$\frac{a}{L}$	Exp	Analytical		F M		Analytical	
		Const	Free	Const	Free	Const[a]	Const[b]
0.0	79.83	82.04	82.04	82.12	82.12	82.21	82.21
0.2	77.79	81.39	68.78	79.80	67.72	80.94	80.56
0.4	75.13	78.10	59.44	75.56	57.02	77.21	75.87
0.6	66.96	71.16	51.18	67.13	48.12	70.27	67.60
0.8	48.34	62.12	43.86	57.06	40.59	61.55	57.89

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Table 4. Fundamental natural frequency (Hz), Interface 3

Shen and Grady [4]				Present Study			
$\frac{a}{L}$	Exp	Analytical		FEM		Analytical	
		Const	Free	Const	Free	Const[a]	Const[b]
0.0	79.83	82.04	82.04	82.12	82.12	82.21	82.21
0.2	80.13	81.46	75.14	81.00	78.32	81.65	81.61
0.4	79.75	79.93	70.42	79.70	73.96	80.10	80.00
0.6	76.96	76.71	65.06	76.69	68 58	76.98	76.77
0.8	72.46	71.66	59.13	72.00	62.10	72.31	71.99

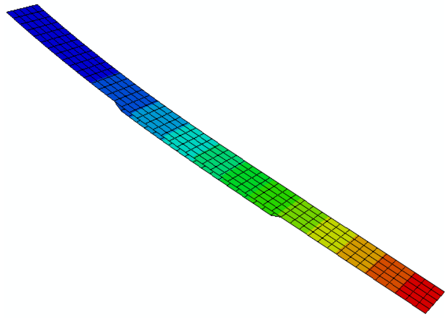
[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Table 5. Fundamental natural frequency (Hz), Interface 4

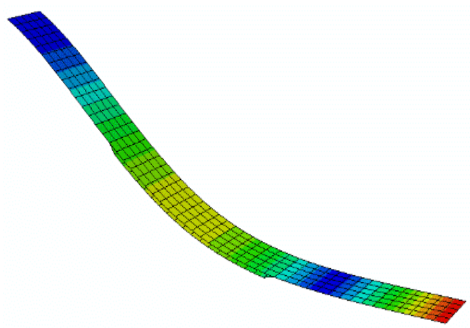
Shen and Grady [4]				Present Study			
$\frac{a}{L}$	Exp	Analytical		EM		Analytical	
		Const	Free	Const	Free	Const[a]	Const[b]
0.0	79.83	82.04	82.04	82.12	82.12	82.21	82.21
0.2	79.96	81.60	75.83	80.88	79.75	81.84	81.70
0.4	68.92	80.38	71.88	79.70	76.63	80.72	80.34
0.6	62.50	77.70	67.18	76.96	72.28	78.33	77.60
0.8	55.63	73.15	61.70	72.66	66.64	74.59	73.44

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Comparing constrained mode and free mode model, for delaminations located near the beam surfaces, free mode models have better agreement with experimental results. Figures 3 and 4 show the first three mode shapes for constrained and free mode models respectively. Delamination dimensionless length (a/L) is 0.4 and is located along interface 4. Blue elements show points with no deflection and refer to nodes of mode shapes. Furthermore, the second and third frequencies obtained from finite element analysis and analytical results from basic function formulation are presented in Tables 6-9. Comparing constrained mode and free mode model, and according to first three mode shapes, when delamination locates on the node of one mode shapes, the discrepancy decreases between related free mode and constrained mode frequencies.



(a)



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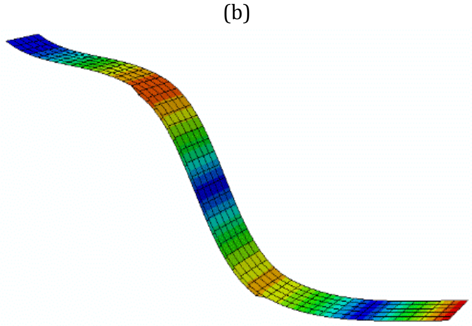
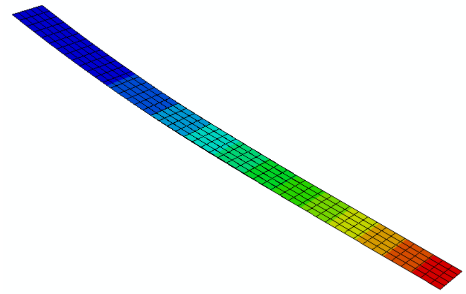
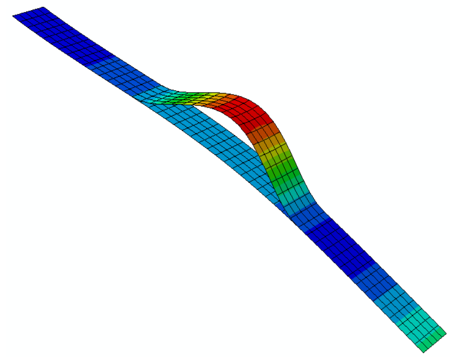


(c)

Figure 3. First three mode shapes of delaminated beam (Interface 4 and a/L=0.4) for constrained mode model
a) 1st mode, *b*) 2nd mode shape, *c*) 3rd mode



(a)



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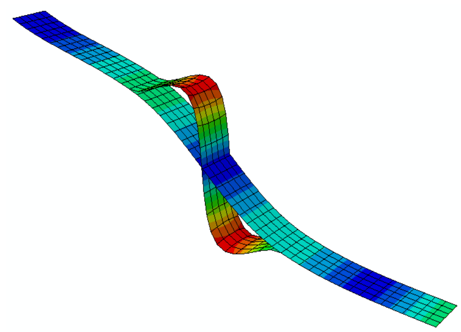

(c)

Figure 4. First three mode shapes of delaminated beam (Interface 4 and a/L=0.4) for free mode model
a) 1st mode, *b*) 2nd mode shape, *c*) 3rd mode

Table 6. 2nd and 3rd natural frequencies (Hz) for analytical delamination along interface 1.

$\frac{a}{L}$	2nd mode				3rd mode			
	FEM		Analytical		FEM		Analytical	
	Const	Free	Const[a]	Const[b]	Const	Free	Const[a]	Const[b]
0.0	514.02	514.02	515.26	515.26	1437.00	1437.00	1449.89	1449.89
0.2	488.99	348.73	490.31	487.13	1216.50	1217.40	1257.99	1233.16
0.4	485.28	313.94	462.55	455.82	851.87	852.73	909.21	872.53
0.6	426.48	284.14	406.94	395.30	752.00	746.88	802.87	770.64
0.8	332.95	248.31	332.18	318.28	720.38	677.55	768.85	734.03

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Table 7. 2nd and 3rd natural frequencies (Hz) for delamination along interface 2.

	2nd mode				3rd mode			
$\frac{a}{L}$	FEM		Analytical		FEM		Analytical	
	Const	Free	Const[a]	Const[b]	Const	Free	Const[a]	Const[b]
0.0	514.02	514.02	515.26	515.26	1437.00	1437.00	1449.89	1449.89
0.2	480.65	342.71	495.00	489.24	1229.50	1231.30	1289.91	1251.77
0.4	481.85	310.50	472.02	460.49	870.19	871.55	963.05	899.63
0.6	431.19	284.06	423.33	403.69	767.48	763.55	851.24	794.39
0.8	341.09	250.47	352.41	328.43	735.51	693.55	820.74	759.63

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Table 8. 2nd and 3rd natural frequencies (Hz) for delamination along interface 3.

	2nd mode				3rd mode			
$\frac{a}{L}$	FEM		Analytical		FEM		Analytical	
	Const	Free	Const[a]	Const[b]	Const	Free	Const[a]	Const[b]
0.0	514.02	514.02	515.26	515.26	1437.00	1437.00	1449.89	1449.89
0.2	493.59	441.50	504.98	504.13	1372.40	1374.50	1383.79	1382.04
0.4	497.61	378.81	494.66	493.25	1172.10	1102.00	1191.86	1186.15
0.6	483.71	451.89	472.89	470.99	1064.00	1156.40	1083.32	1076.74
0.8	438.10	423.84	430.14	427.71	1043.10	1060.30	1061.96	1055.06

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Table 9. 2nd and 3rd natural frequencies (Hz) for delamination along interface 4.

	2nd mode				3rd mode			
$\frac{a}{L}$	FEM		Analytical		FEM		Analytical	
	Const	Free	Const[a]	Const[b]	Const	Free	Const[a]	Const[b]
0.0	514.02	514.02	515.26	515.26	1437.00	1437.00	1449.89	1449.89
0.2	490.79	465.59	508.85	505.69	1377.90	1381.60	1397.59	1392.17
0.4	495.86	413.08	501.64	496.41	1192.80	1152.30	1239.71	1220.29
0.6	485.50	451.20	484.34	477.50	1088.90	1141.50	1140.50	1116.76
0.8	444.67	423.11	447.88	439.25	1069.00	1078.70	1121.30	1096.38

[a] Without bending-extension coupling.
[b] With bending-extension coupling.

Also, analytical frequencies of delaminated beam with bending-extension coupling assumptions decrease more rather than without bending-extension coupling condition.

Here, first three natural frequencies are presented for single delamination located at various lengthwise locations. Tables 10 and 11 show results for beams with clamped-free and clamped-clamped boundary conditions respectively. Delamination length (a/L) is 0.2 and is located in interface 1 (mid-plane) through thickness for all cases.

Natural frequencies from constrained mode model of finite element analysis better match the experiments, for all cases. Also for small delamination lengths (a/L<0.3) there is no considerable difference between analytical frequencies obtained from both constrained and free mode model, especially for lower natural frequencies. Therefore, only results of constrained mode model with bending-extension coupling are presented here.

5. Conclusion

In this study, a new simple mathematical technique is proposed for free vibration analysis of a delaminated composite beam with any numbers of delaminations. This technique is verified for a composite beam with single delamination. Therefore, a delaminated beam is divided into four Euler-Bernoulli beams and beams on both sides of delamination have free or constrained modes. The effects of delamination size and location on vibration characteristics of laminated composite beams are investigated. Modal parameters are extracted using both analytical and finite element methods. In the analytical part, the expressions for displacement functions are written based on basic standard trigonometric and hyperbolic functions for simplifying the calculation. In the finite element part, constrained mode and free mode models are simulated using suitable elements, boundary conditions, interactions and constrains. The numerical results are validated by the analytical and experimental results available in the literature for some case studies. These results of the proposed analytical method and finite element method show good agreement with them and with each other.

The numerical results show that delamination in composite beams affects the natural frequencies and this change depends not only on the size of delamination but also on its location. Comparing constrained mode and free mode model, the natural frequencies obtained from free mode model are comparatively lower than the results of constrained mode model. This discrepancy can be attributed to the effect of contact between the delaminated surfaces during vibrations. Also, for delaminations located near the beam surfaces, free mode models have better agreement with them and with experimental results. Vice versa, when delamination moves from the surface to the mid-plane of beam, constrained mode frequencies better match the experiment. Also according to corresponding mode shapes, when delamination locates on the node of one mode shapes, the discrepancy decreases between related free mode and constrained mode frequencies.

Natural frequencies from constrained mode model of finite element analysis better match the experiment, for all cases. Also, for small delamination lengths (a/L<0.3) there is no considerable difference between analytical frequencies obtained from both constrained and free mode model, especially for lower natural frequencies.

Finally, the obtained results can be used as a database to predict the frequency behaviour of a delaminated beam by changing the size and location of delamination. The results show that if the effect of the delamination can be accurately predicted, then changes in modal parameters can be used to determine the location and extent of delamination in composite structure.

Table 10. First three natural frequencies (HZ) for Clamped-free boundary conditions

$\frac{a_l}{L}$	1st mode		2nd mode		3rd mode	
	FEM	Anal	FEM	Anal	FEM	Anal
Intact	82.12	82.21	514.02	515.26	1437.00	1442.89
0.15	77.04	75.73	450.10	455.55	1202.00	1214.73
0.50	80.17	80.41	488.99	487.13	1216.50	1233.16
0.85	81.89	82.09	469.94	499.50	1308.80	13 5.21

Table 11. First three natural frequencies (HZ) for Clamped-clamped boundary conditions

$\frac{a_l}{L}$	1st mode		2nd mode		3rd mode	
	FEM	Anal	FEM	Anal	FEM	Anal
Intact	521.79	523.19	1435.50	1442.23	2807.40	2827.35
0.15	463.33	465.57	1209.70	1212.91	253 .00	2486.32
0.30	492.35	490.06	1467.80	1377.78	2313.00	2333.49
0.50	531.10	503.39	1229.10	1245.58	2876.20	2661.52

Nomenclature

L	beam length
h	beam thickness
b	beam width
A	delamination length
D	reduced bending stiffness
ρ	mass density
A	cross-sectional area
A_{11}	extensional stiffness
B_{11}	coupling stiffness
D_{11}	bending stiffness
ω	natural frequency
W	mode shape
λ	non-dimensional frequency

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