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## Dynamic Stability of Moderately Thick Composite Laminated Skew Plates using Finite Strip Method

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### ABSTRACT

The dynamic instability regions of composite laminated skew flat plates subjected to uniform in-plane axial end-loading are investigated. The in-plane loading is assumed as a combination of a time-invariant component and a harmonic time-varying component uniformly distributed along two opposite panel ends' width. In case of some loading frequency-amplitude pair-conditions, the model is subjected to instabilities. The dynamic instability margins of the skewed flat panel have been extracted using a developed semi-analytical finite strip formulation. The method has been developed based on a full-energy approach through the principle of the virtual work. The effects of thickness have been included by utilizing a third-order Reddy type shear deformation theory. The effects of boundary conditions as well as geometry on the instability load-frequency regions are derived using the Bolotin's first-order approximation. In order to demonstrate the capabilities of the developed method in predicting the structural dynamic behavior, some representative results are obtained and compared with those in the literature wherever available.

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## 1. Introduction

The thin-walled structures especially those made of the laminated composite materials are widely used in air, space and marine structures. These structural products are advantageous due to their least structural weight whilst providing the required strength. Periodic time-varying loads are prevalent in case of the pre-mentioned mechanical structures as well as fluid-structural interactions. A thin-walled panel, under in-plane dynamic loading where a constant mean load is added to a harmonically varying excitation with a constant frequency, meets situations where the instability conditions may appear even while the amplitude of the corresponding dynamic instability load is not exceeding the value corresponding to the panel static bifurcation point. The panels of non-rectangular plan-form such as skew plates find wide application as mechanical structures.

Heuer et al. (1993) [1] studied the nonlinear random vibrations characteristics of thermally

buckled skew plates. Using the Galerkin-procedure follows by assuming an effective white noise excitation, the probability of first occurrence of dynamic snap-through has been determined. Hu and Tzeng (2000) [2] used a finite element method approach to investigate the static buckling phenomenon in skew plates subjected to the uniaxial compressive in-plane loads. The ABAQUS software was used and the effects of skew angles, laminate layups, plate aspect ratios, plate thicknesses, central circular cutouts, and end conditions on the buckling of skew composite laminate plates were presented. Dey and Singha (2006) [3] considered the simply supported skew plate subjected to periodic in-plane loads and extracted the dynamic instability regions of the structure. The finite element approach including the effects of transverse shear deformation, in-plane and rotary inertia has been utilized. The Bolotin's first and second-order approximation has been implemented in order to extract the boundaries of the instability regions for various skew angles, thick-

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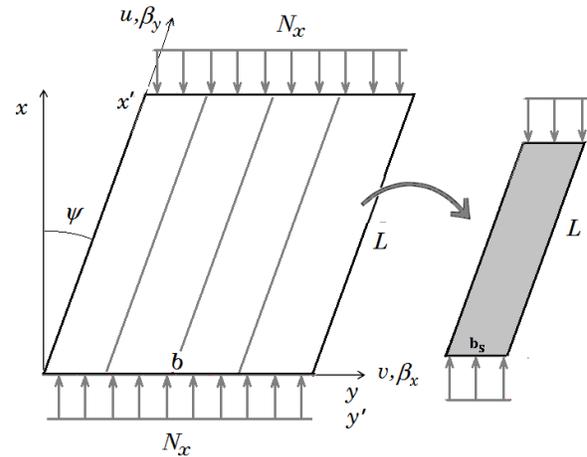
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ness-to-span ratios, fiber orientations and static in-plane loadings. Lee (2010) [4] used the finite element method to investigate dynamic stability of laminated skew plates. The skew structures were assumed to be subjected to in-plane pulsating forces. A higher order shear deformation plate element and Bolotin's approximation was utilized. These structures also assumed to have cutout zones and the effect of different geometry and loading parameters was studied. Tahmasebinejad and Shanmugam (2011) [5] conducted the problem of elastic buckling behavior of uniaxially loaded skew plates with openings subjected to uniaxial longitudinal compression. Circular and skew shape cutouts of different sizes have been considered through the application of finite element software package ABAQUS. The effectiveness of skew angle, size, shape and position of openings and aspect ratio of the plates on the buckling critical load has been examined. Noh and Lee (2014) [6] carried out the dynamic instability problem of delaminated composite skew plates subjected to in-plane periodic loadings. The effect of various parameters on the dynamic stability of delaminated composite skew structures has been investigated. The formulation was based on the higher order plate theory and the Bolotin's approximation.

In the current study the dynamic stability of laminated skew flat panels subjected to the uniform in-plane end-loading has been investigated through the implementation of finite strip formulations. The loading is assumed to change harmonically with time. The problem is formulated using a developed semi-analytical finite strip method. The formulation is based on the classical plate and shell theory while the Reddy type higher-order theory is also used in order to include the transverse shear stresses effect in case of the moderately thick structures. The governing equations are derived using the full energy concepts on the basis of the virtual work principal. The instability load frequency regions corresponding to the assumed in-plane parametric loading are derived utilizing the Bolotin's first-order approximation. The equations are extracted using eigen solution algorithm and some representative problems are studied.

## 2. Theoretical Development

The structure geometry is assumed to be composed of a series of longitudinal skewed strips. Fig. 1 depicts a general skew plate model with finite strips of skew angle  $\psi$ , length  $L$ , width  $b_s$  and thickness  $t$ . Two nodal lines exist in the transverse direction of every strip element.



**Figure 1.** The loading and geometry of a typical skew plate with the longitudinal finite strips

Strips are subjected to uniform time varying longitudinal in-plane loading at their two ends which is assumed to result in a corresponding uniform unidirectional time-varying stress field throughout the strip's area. The loading is assumed to consist of a constant (static) and harmonically time-varying (dynamic) components, which are identified with S and D superscript signs, respectively. These load component coefficients can be expressed as fractions of the critical static buckling load of the structure,  $N_{cr}$ . Thus, the following equation is derived:

$$N_x = a^S N_{cr} + a^D N_{cr} \cos \omega t \quad (1)$$

where  $\omega$ ,  $a^S$  and  $a^D$  are the excitation frequency, static load fraction coefficient and dynamic load fraction coefficient, respectively. The static and dynamic components of the end-loading are considered uniform throughout the whole panel area.

The solution of the problem of parametric instability is sought through the principle of virtual work. The total energy of a strip is defined with kinetic ( $T$ ), pre-stress ( $U_g$ ) and strain ( $U$ ) energy components as what follows:

$$\Pi = U - U_g - T \quad (2)$$

The kinetic energy of a flat strip with uniform density can be found through the following equation:

$$T = \frac{\rho t}{2} \iint \dot{u}^{\circ 2} + \dot{v}^{\circ 2} + \dot{w}^{\circ 2} + \frac{t^2}{12} \dot{\beta}_x^2 + \dot{\beta}_y^2 \, dx dy \quad (3a)$$

The pre-stress strain energy can be calculated using the following equation:

$$U_g = \frac{t}{2} \iint N_x \left[ u_{,x}^{\circ 2} + v_{,x}^{\circ 2} + w_{,x}^{\circ 2} + \frac{t^2}{12} \beta_{x,x}^2 + \beta_{y,x}^2 \right] dx dy \quad (3b)$$

And the strain elastic energy can be found using the following equation:

$$U = \frac{1}{2} \iint \left\{ \langle N M O P \rangle \cdot \langle \varepsilon^{(0)} \varepsilon^{(1)} \varepsilon^{(2)} \varepsilon^{(3)} \rangle^T + \langle Q R T U \rangle \cdot \langle \gamma^{(0)} \gamma^{(1)} \gamma^{(2)} \gamma^{(3)} \rangle^T \right\} dx dy \quad (3c)$$

where  $\rho$  is the material mass density and ' $\dot{\phantom{x}}$ ' superscript represents the time differentiation. The force resultants (i.e.  $N, M, O, P, Q, R, T, U$ ) could be calculated using composite layered stiffness matrices through the following equations,

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{O\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [D] & [E] \\ [B] & [D] & [E] & [F] \\ [D] & [E] & [F] & [I] \\ [E] & [F] & [I] & [H] \end{bmatrix} \begin{Bmatrix} \{\epsilon^{(0)}\} \\ \{\epsilon^{(1)}\} \\ \{\epsilon^{(2)}\} \\ \{\epsilon^{(3)}\} \end{Bmatrix}, \tag{4}$$

$$\begin{Bmatrix} \{Q\} \\ \{R\} \\ \{T\} \\ \{U\} \end{Bmatrix} = \begin{bmatrix} [A^S] & [B^S] & [D^S] & [E^S] \\ [B^S] & [D^S] & [E^S] & [F^S] \\ [D^S] & [E^S] & [F^S] & [I^S] \\ [E^S] & [F^S] & [I^S] & [H^S] \end{bmatrix} \begin{Bmatrix} \{\gamma^{(0)}\} \\ \{\gamma^{(1)}\} \\ \{\gamma^{(2)}\} \\ \{\gamma^{(3)}\} \end{Bmatrix}$$

Substituting the energy terms (Eq. (3)) into Eq.(2), using Eq. (4) and applying the principle of virtual work, factorizing with respect to degrees of freedom vectors and building matrices, assembling the strip matrices and implementing the necessary boundary conditions, a system of Mathieu type differential equations is obtained as follows:

$$M\ddot{\delta} + (K - a^S K_g^S - a^D \cos \omega t K_g^D) \delta = 0 \tag{5}$$

Where  $M, K, K_g^S$  and  $K_g^D$  are the square global structural matrices corresponding to the mass, strain, static and dynamic initial stress energies, respectively.  $\delta$  is the global vector of unconstrained degrees of freedom. Whenever initial stress state is ignored, eq. (5) reduces to a free vibration problem whilst in case of excluding the effects of time and mass, with the presence of initial stress, the problem changes to an eigen value static buckling one.

By implementation of Bolotin's first-order approximation corresponding to twice the loading period ( $2P$ ), Eq. (5) is changed to a set of eigen-value problems as follow:

$$\begin{cases} K - a^S K_g^S + \frac{1}{2} a^D K_g^D - \frac{1}{4} \omega^2 M = 0 \\ K - a^S K_g^S - \frac{1}{2} a^D K_g^D - \frac{1}{4} \omega^2 M = 0 \end{cases} \tag{6}$$

Solving the abovementioned eigenvalue system with respect to the three unknown parameters (i.e.  $a^S, a^D, \omega$ ) results in two stability bounds in the space of loading frequency- loading amplitude.

In order to include the effects of through the thickness transverse shear strains as parabolic variation functions of the thickness dimension, a Higher-order Shear deformation theory (HST) of Reddy's type is assigned for the displacement field approximation. The strip displacement field at any arbitrary point can be expressed as follows:

$$\begin{cases} u(x, y, z) = u^0(x, y) + z\beta_x + z^3\beta_x^* \\ v(x, y, z) = v^0(x, y) + z\beta_y + z^3\beta_y^* \\ w(x, y, z) = w^0(x, y) \\ \beta_x^* = -4(w_x^0 + \beta_x) / 3t^2 \\ \beta_y^* = -4(w_y^0 + \beta_y) / 3t^2 \end{cases} \tag{7}$$

The linear strains at the mid-surface of the skew strip can be simply expressed as follows:

$$\begin{cases} \epsilon_x = u_{,x} \\ \epsilon_y = v_{,y} \end{cases} \begin{cases} \gamma_{xy} = u_{,y} + v_{,x} \\ \gamma_{yz} = v_{,z} + w_{,y} \\ \gamma_{xz} = u_{,z} + w_{,x} \end{cases} \tag{8}$$

The strip's mid-plane displacement approximating functions are composed of a multiplication of approximating terms in the longitudinal as well as the transverse directions (eq. 9). In the longitudinal directions, the trigonometric functions are utilized while in the transverse direction, Lagrangian as well as Hermitian functions are employed for in-plane and out of plane displacements, respectively.

$$\begin{Bmatrix} u^0 \\ v^0 \\ w^0 \\ \beta_y \\ \beta_x \end{Bmatrix} = \sum_{k=1}^n \begin{bmatrix} T_u & & & & \\ & T_v & & 0 & \\ & & T_w & & \\ & & & T_{\beta_y} & \\ & & & & T_{\beta_x}(x) \end{bmatrix} \cdot \begin{Bmatrix} d^u \\ d^v \\ d^w \\ d^{\beta_y} \\ d^{\beta_x} \end{Bmatrix} \tag{9}$$

where  $T(x)$  is longitudinal displacement functions that satisfies the kinematic conditions prescribed at the strip ends and the parameter  $n$  represents, the number of longitudinal terms is kept equal to three in the current study (i.e.  $n=3$ ). The boundary conditions at the two loaded ends are assumed here to be of simply supported type, thus cosine function for  $u^0, \beta_x$  and sine function for the other functions are presumed. In order to support other kinds of end conditions, different trigonometric functions could be utilized. It is also to be noted that the strip side lines can accommodate any kind of boundary conditions, namely simply supported, clamped or free.

Oblique coordinate system (Fig.2) is used where the skew angle,  $\psi$ , is measured in relation to the longitudinal direction. The transformation relationships between the global coordinate system  $(x,y,z)$  and the oblique coordinate system  $(x',y',z')$  are as follows:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos \psi & 0 & 0 \\ \sin \psi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \tag{10}$$

The first and second derivatives of any typical function  $f(x,y)$  maybe expressed in the term of normal coordinate as a function of the skew angle as follows [7],

$$\begin{Bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{Bmatrix} = \begin{bmatrix} \sec \psi & -\tan \psi \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \partial f / \partial x' \\ \partial f / \partial y' \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \partial^2 f / \partial x^2 \\ \partial^2 f / \partial y^2 \\ \partial^2 f / \partial x \partial y \end{Bmatrix} = \begin{bmatrix} \sec^2 \psi & \tan^2 \psi & \sec \psi \tan \psi \\ 0 & 1 & 0 \\ 0 & -2 \tan \psi & \sec \psi \end{bmatrix} \begin{Bmatrix} \partial^2 f / \partial x'^2 \\ \partial^2 f / \partial y'^2 \\ \partial^2 f / \partial x' \partial y' \end{Bmatrix}$$

### 3. Results and Discussions

A semi-analytical based finite strip numerical code is developed based on the HST assumptions. Some case studies are studied and the results are presented in this section.

Large amplitude free flexural vibration characteristics of moderately thick laminated composite skew plates are studied. The material properties, unless specified otherwise, used in the present analysis are as follows:

$$\begin{cases} E_1 / E_2 = 40 \\ G_{12} / E_2 = G_{13} / E_2 = 0.6, G_{23} / E_2 = 0.5 \\ \nu_{12} = 0.25 \end{cases} \quad (12)$$

Subscripts  $_1$  and  $_2$  represent the longitudinal and transverse directions with respect to the fiber orientation, respectively. All layers are assumed to be of equal thickness. The fiber orientation is measured from longitudinal axis. The plate is opposed to simply supported conditions on all its boundaries. The plate is assumed moderately thick with  $L/t$  ratio of 10. The cross-ply layup  $[90/0/90/0/90]$  and variable skew angles are considered.

Table 1 presents the non-dimensionalized first five natural frequencies of the skew plate from the First-order Shear deformation plate Theory (FST) data available in the literature as well as HST sa-FSM calculated results. The results show that the HST-FSM predicts lower and more conservative frequencies related to the FST results.

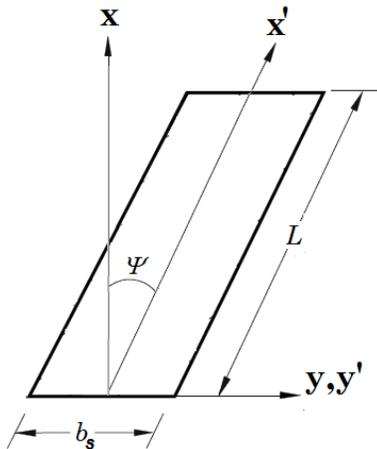


Figure 2. The normal and oblique local coordinates systems of a typical skew finite strip

The principal region of the dynamic instability of simply supported 8-layered cross-ply  $[90/0]_2$  square as well as 30degrees-skew plates ( $L=b, L/t=10, \alpha^s=0$ ) are presented in Fig. 3. The results from the FST finite element method analysis is also provided for the sake of comparison. The results show a good consistency of prediction of instability regions.

A symmetric crossply laminated  $[90/0]_s$  panel is considered using material properties of eq.(12) and geometrical characteristics of  $L/t=10$  and  $b/L=1$ . The simply supported boundary conditions are considered. A number of sensitivity analyses are made on different parameters including the panel aspect ratio, panel skew angle, panel layup and side boundary conditions. The static loading coefficient is assumed to be zero. A FSM model with 10 strips is utilized to extract results.

First, the simply supported square as well as 10 degrees skew panel with three aspect ratios of  $b/L=0.5, 1.0$  and  $2.0$  are studied.

Table 1. The dimensionless fundamental natural frequency ( $\Omega = \omega L^2 / \pi^2 t \sqrt{\rho / E_2}$ )

Mode	FST [8]	FST FEM [7]	HST FEM [4]	HST sa-FSM
No skew				
1	1.5699	1.5700	1.5701	1.5531
2	3.0371	3.0386	2.9034	2.9656
3	3.7324	3.7422	3.7813	3.7101
4	4.5664	4.576	4.6212	4.5066
5	5.1469	5.1667	6.2438	4.9676
Skew angle = 30				
1	2.0844	2.0805	2.0366	2.0634
2	3.5127	3.5491	3.3581	3.4713
3	4.6997	4.7107	4.6084	4.7038
4	4.8855	4.9435	4.6943	4.9167
5	6.2494	6.4200	5.8051	6.3376
		196 elements		10 strips 180 DOF

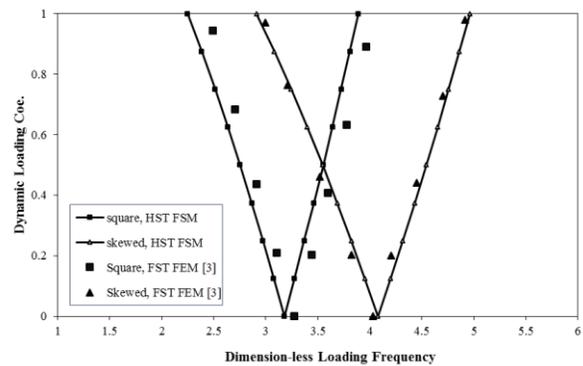


Figure 3. The dynamic stability boundaries of thick 8-layered crossply composite plate with and without skewness

The resulting instability regions presented in Fig. 4 show that the higher the aspect ratio is, the narrower and more critical the instability region is. Also, it is shown that the skew panel instability occurs at higher instability frequency ranges for all aspect ratio case studies.

Simply supported panel with different skew angles is studied. Fig. 5 depicts the change in the fundamental instability region with increased skew angle of the panel while Figs. 6 and 7 represent the changes in the first three natural frequencies as well as many first dimension-less static buckling strengths of panels. The results show that the higher skew angles shift the instability loading frequencies to the higher ones but lead to the wider instability regions (wider frequency intervals). The results also show that an increase in the skew angle causes slight increase in both natural frequency and buckling strength of the panel. There is an exception here for the second mode buckling strength where it is not followed by the rate of change in the other modes and meets very small changes.

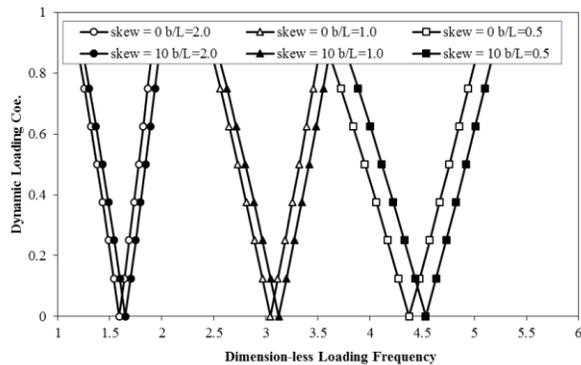


Figure 4. The effect of skew angle and aspect ratio on the dynamic instability regions of the flat panel

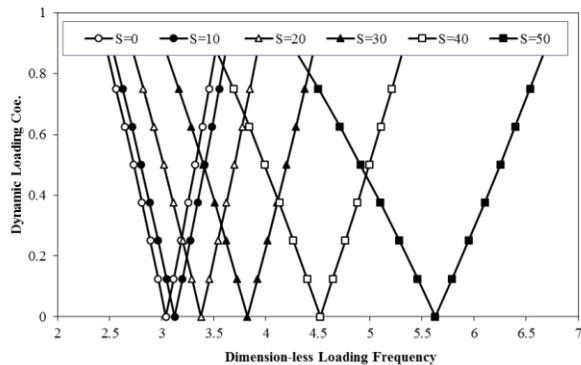


Figure 5. The effect of skew angle on the dynamic instability regions of the flat panel

The square as well as 10 degrees skew laminated simply supported panel with three different layups are considered. The 2, 4 and 8 layer anti-symmetric crossply laminates are investigated while the  $L/t$  ratio is kept fixed. As shown in Fig. 8, the 2-layer model presents more critical instability region that is due to its stronger anti-symmetry effects. For other layups, more layers means a right shifted instability region with higher instability frequencies. The skew panels show a safer instability region with slightly higher instability ranges with respect to the square geometries.

The symmetric laminated 10 degrees skew panel with the two loaded ends simply supported are considered. The two side boundary conditions have altered between clamped, free and simply supported ones. Fig. 9 presents the resulting instability regions where it clearly shows the stabilizing effects of tighter boundary constraints on the parametric behavior. The panels under the clamped side boundaries reveal higher instabilizing frequencies with narrower regions with respect to the simply supported ones.

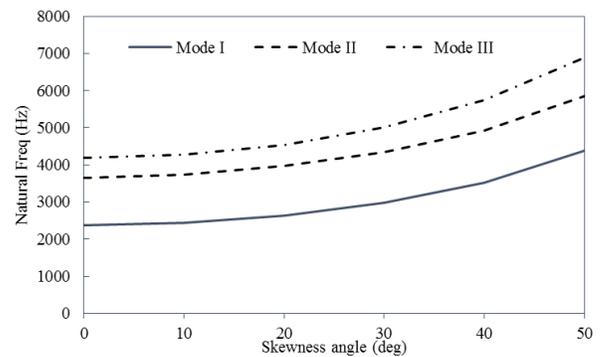


Figure 6. The effect of the skew angle on the natural frequency of the flat panel

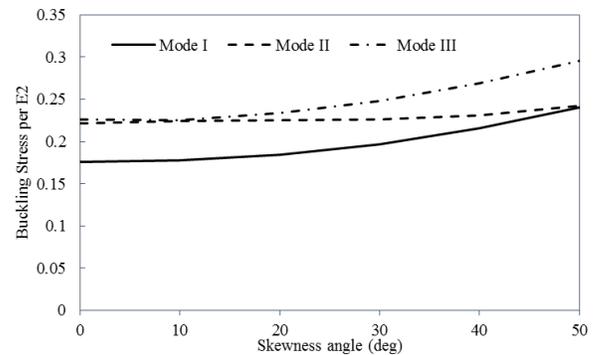


Figure 7. The effect of the skew angle on the buckling strength of the flat panel

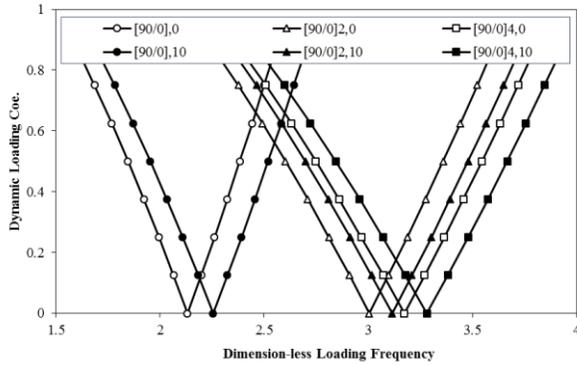


Figure 8. The effect of the skew angle and the layup on the dynamic instability regions of the flat panel

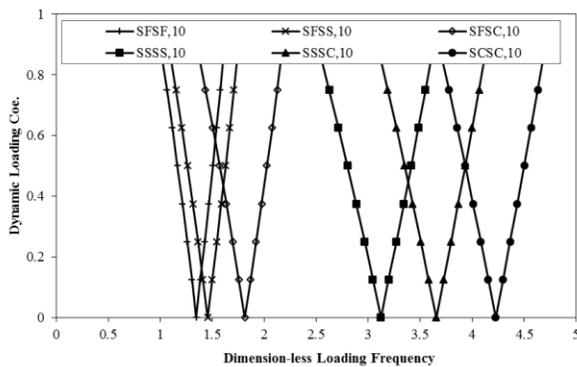


Figure 9. The effect of the boundary conditions on the dynamic instability regions of the skew flat panel

#### 4. Conclusions

The dynamic instability regions of the composite laminated skew plates subjected to the uniform in-plane longitudinal end-loading are investigated. The dynamic instability margins of the skewed flat panel have been extracted using the developed semi-analytical finite strip formulation. The effects of the thickness have been included by utilizing a third-order Reddy type shear deformation theory. The effects of boundary conditions as well as the static loading on instability load-frequency regions are derived using the Bolotin's first-order approximation. In order to demonstrate the capabilities of the developed method in predicting the structural dynamic behavior, the obtained results are compared with those in the literature wherever available. It is shown that the sa-FSM with three approximation

functions (i.e.  $n=3$ ) is capable of precisely predicting the instability of thick skew plates. It is to be added that the accuracy of the method could be enhanced through using more approximating series terms (i.e. greater  $n$ ,  $p$  version convergence) and more refined mesh (i.e.  $h$  version convergence). Also, the stabilizing effect of higher skew angle, higher laminate counts, and lower aspect ratios was shown on the parametric behavior of flat plates under harmonic longitudinal in-plane loadings.

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