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A Semi Analytical Nonlinear Approach For Size-Dependent Analysis of Longitudinal Vibration in Terms of Axially Functionally Graded Nanorods

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KEYWORDS

Nonlinear longitudinal vibration;
Nanorod;
Functionally graded materials;
nonlocal elasticity theory.

ABSTRACT

In this paper, size dependency of the nonlinear free longitudinal vibration of axially functionally graded (AFG) nanorods is studied using the nonlocal elasticity theory. A power-law distribution is considered for the variations of the nanorod material properties through its length. The nonlinear equation of motion including the von-Karman geometric nonlinearity is derived using Hamilton's principle. Then, a solution for the linear equation of motion is obtained using the harmonic differential quadrature method, and mode shapes and natural frequencies are extracted. At the next step, the nonlinear natural frequencies are calculated by solving the nonlinear equation of motion using the multiple scales method. Two types of boundary conditions are considered, i.e. fixed-fixed and fixed-free. The presented results include effects of various parameters like nanorod length and diameter, amplitude of vibration, small scale parameter, and frequency number, on the natural frequencies. In addition, a comparative study is conducted to evaluate effects of the type of linear mode shape on the nonlinear natural frequencies.

1. Introduction

The experimental and theoretical studies on the various behaviors of nano-sized structures have shown that they have superior properties in comparison with the structures at macro-scale. These superior properties can be enhanced by using functionally graded materials (FGM) for nano-sized structures. FGMs have been used in the nano-technology field to attain desired efficiency [1-4]. Since behaviors of nano-sized FGM structures are different from those made of homogeneous materials, it is incumbent to be intinate with the behaviors of FGM nano-structures for NEMS/MEMS fabrication.

One of the nanoscale elements that attracted the attention of researchers is the nano-beam. There are many works regarding analysis of its various mechanical behaviors, buckling [5, 6], postbuckling [7, 8], bending [9, 10], stability [11, 12], free and forced axial vibration [13, 14], free and forced torsional vibration [15, 16], free and forced transverse vibration [17-20], and wave propagation [21, 22]. Among the studies, axial behavior of nanobeams (nanorods) has more recently been investigated. These investigations have been done based on three types of size-dependent theories, surface elasticity theory, strain gradient theory, and nonlocal elasticity theory. Free vibration, wave propagation, and instability of nanorods/nanotubes are investigated using the nonlocal elasticity theory. In these researches effects of various factors like elastic medium [23-27], magnetic fields [27-30], the shape of nanorod cross section (uniform [31, 32], non-uniform [33, 34], tapered [35, 36], cone-like [37]), crack [25, 38, 39], discontinuity [40], type of rod theory [26, 41-45], number of coupled nanorods [13, 30, 46-49], number of nanotube walls [23, 50-52], and type of nanorod material properties [53-56] have been considered. The other studies in the field of analysis of longitudinal behavior of nanorods/nanotubes are done based on the surface elasticity and strain gradient theories. Free vibration of homogeneous [57, 58], axially functionally graded [59-61], viscoelastic [62, 63], and embedded nanorods [64] are analyzed using the strain gradient theory. Finally using the surface elasticity theory, Nazemnezhad and Shokrollahi

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have considered free longitudinal oscillation of FGM nano-sized rods [65].

All of the mentioned works have only analyzed linear axial behavior of nanorods/nanotubes based on different theories. There are only a few works studying the nonlinear axial vibration of nanotubes/nanorods. Nazemnezhad et al. [66] studied the nonlinear longitudinal oscillation and internal resonances of nanoscale rods incorporating effects of the surface energy parameters. In another work, the CNTs had been modeled as a homogeneous nanorod, and their nonlinear free axial vibration was investigated [67]. Fatahi-Vajari and Azimzadeh [68] considered the nonlinear axial vibration of single-walled carbon nanotubes based on Homotopy perturbation method. In this research, the nonlinear governing equation of motion was derived using doublet mechanics. The nonlinear coupled axial–torsional vibration of single-walled carbon nanotubes had also been analyzed [69]. In this study, Galerkin method and homotopy perturbation method were used. In another different work, the internal resonances of nanorods were studied based on the nonlocal elasticity theory [70]. In this work, Nazemnezhad and Zare implemented the multi-mode Galerkin method to convert the partial differential equation to the ordinary differential type. Jamali Shakhlavi et al. [71] had studied similar research on internal resonances of Rayleigh nanorods. Finally, Jamali Shakhlavi et al. [72] considered the effects of thermal and magnetic fields and elastic medium on nonlinear axial vibration of nanorods. The nonlocal elasticity was used and the von Kármán geometric nonlinearity was taken into account.

As mentioned in the first paragraph of this section, using functionally graded materials enhances the mechanical properties of structures. In addition, the literature survey done showed that there is a lack in investigating the nonlinear axial vibration of nanotubes/nanorods. The lack is what the effect of material on the nonlinear axial behavior of nanotubes/nanorods is. Therefore, this study presents an investigation of the nonlinear axial behavior of nanorods/nanotubes when they are made of FGMs. To this end, a comprehensive model is proposed to investigate the nonlocal parameter effect on the large amplitude oscillation of nano-size axially functionally graded (AFG) rods with von Kármán type nonlinearity. In this regard, the equation of motion and corresponding boundary conditions are obtained. Then, linear mode shapes and natural frequencies are extracted using the harmonic differential quadrature method. In the next step, using the multiple scales method, the nonlinear governing equation of motion is solved. Finally, the nonlocal parameter effects on the frequencies of AFG nanorods are investigated for various end conditions, nanorod lengths, nanorod diameter, and amplitude ratios.

2. Problem Formulation

2.1. Governing Equation of Motion and Corresponding Boundary Conditions

In order to derive the nonlinear governing equation of motion and corresponding boundary conditions of AFG nano-size rods, an AFG nano-size rod with length L ($0 \leq x \leq L$), cross-sectional area A and diameter D is desired in xyz coordinates as shown in Figure 1.

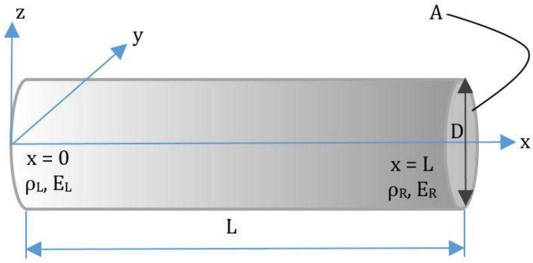

Fig. 1. Schematic of the AFG nanorod

Two different materials at the left and right ends constitute the AFG nano-size rod. The Poisson's ratio is assumed to be constant but the shear modulus $G(x)$, bulk elastic modulus $E(x)$, and mass density $\rho(x)$ of the AFG nanorod vary in the longitudinal direction according to the power low distribution as follow [73, 74]:

$$E(x) = (E_L - E_R)\left(\frac{L-x}{L}\right)^m + E_R$$
$$\rho(x) = (\rho_L - \rho_R)\left(\frac{L-x}{L}\right)^m + \rho_R \qquad (1)$$

In Eq. 1, m is the gradient index and shows the profile of the mechanical properties of the AFG nano-size rod across its length, and the superscripts R and L imply the right (x=L) and left (x=0) ends of the AFG nanorod, respectively. In Figure 2 variations of the mass density and the Young's modulus of elasticity across the AFG nanorod length are plotted for various values of the gradient index.

Based on the rod theory, the displacement field at any point of the AFG nano-size rod is given as below:

$$w_1(x,t) = u_0(x,t);$$
$$w_2(x,t) = 0; \qquad (2)$$
$$w_3(x,t) = 0;$$

where t is the time, and $w_1(x,t)$, $w_2(x,t)$ and $w_3(x,t)$ are the displacement components along x, y, and z directions, respectively. To obtain the strain(s) in the AFG nanorod, the von-kármán strain-displacement relation with the nonlinear terms is used

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} + \frac{\partial u_m}{\partial i}\frac{\partial u_m}{\partial j}\right) \qquad (3)$$

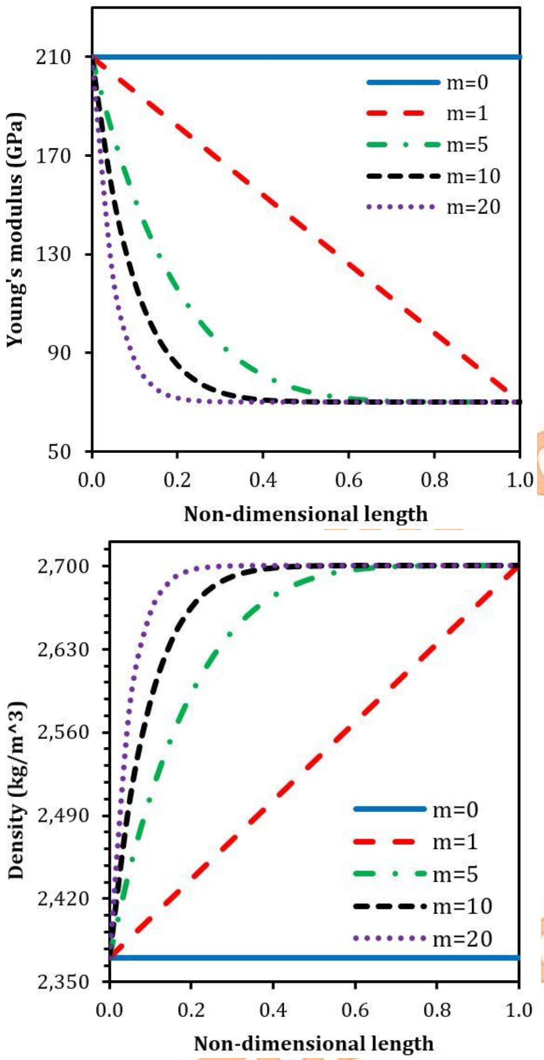

Fig. 2. Variations of the mass density and the Young's modulus across the length of AFG nanorod

According to Eqs. (2)-(3), the non-zero strain in the cross section of the AFG nanorod is obtained as

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2 \tag{4}$$

From the Hooke's law, the stress induced in the cross section of the AFG nanorod is also obtained as

$$\sigma_{xx} = E(x)\varepsilon_{xx} = E(x)\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right) \tag{5}$$

It is now possible to derive the governing equation of motion and the corresponding boundary conditions using Hamilton's principle given as Eq. (6)

$$\delta\int_{t_1}^{t_2} (U - T)dt = 0 \tag{6}$$

where T and U are the kinetic and potential energies of the AFG nano-size rod, respectively and are given as

$$\delta U = \int_V \sigma_{xx}\delta\varepsilon_{xx}dV \tag{7}$$

$$\delta T = \int_V \rho(x)\frac{\partial u_0}{\partial t}\frac{\partial(\delta u_0)}{\partial t}dV \tag{8}$$

Now substituting Eqs. (2), (4)-(5) into Eqs. (7)-(8) results in

$$\frac{\partial}{\partial x}\left(P_{xx}\left(1 + \frac{\partial u_0}{\partial x}\right)\right) - \rho(x)A\frac{\partial^2 u_0}{\partial t^2} = 0 \tag{9}$$

$$P_{xx}\left(1 + \frac{\partial u_0}{\partial x}\right)\delta u_0\Big|_0^L = 0 \tag{10}$$

where P_{xx} is the local force resultant and is described as

$$P_{xx} = \int_0^L \sigma_{xx}dA = E(x)A\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right) \tag{11}$$

Eqs. (9) and (10) are the local governing equation of motion and corresponding boundary conditions, respectively.

In order to consider the nonlocal parameter effect on the governing equation of motion of the AFG nano-size rod, the nonlocal elasticity theory [75] is used. To this end, Eq. (9) can be stated in the nonlocal form as

$$\frac{\partial}{\partial x}\left(P_{xx}^{nl}\left(1 + \frac{\partial u_0}{\partial x}\right)\right) - \rho(x)A\frac{\partial^2 u_0}{\partial t^2} = 0 \tag{12}$$

where subscript *nl* denotes the nonlocal, and P_{xx}^{nl} is the force resultant in nonlocal form or it is the nonlocal force resultant. According to the nonlocal elasticity theory, the P_{xx}^{nl} can be obtained from Eq. (11) as follows

$$(1 - \mu\nabla^2)P_{xx}^{nl} = P_{xx} = E(x)A\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right) \tag{13}$$

In Eq. (13), $\nabla^2 = \frac{\partial^2}{\partial x^2}$ is the one-dimensional Laplacian operator. Now, P_{xx}^{nl} is obtained from Eq. (13) as [70]

$$P_{xx}^{nl} = \frac{1}{(1 - \mu\nabla^2)}\left(E(x)A\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right) \tag{14}$$

By obtaining the nonlocal force resultant, P_{xx}^{nl}, the nonlocal governing equation of motion of the AFG nanorod can be given in terms of displacement by substituting Eq. (14) into Eq. (12) as

$$\frac{\partial}{\partial x}\left(\frac{1}{(1 - \mu\nabla^2)}\left(E(x)A\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right) \times \left(1 + \frac{\partial u_0}{\partial x}\right)\right) - \rho(x)A\frac{\partial^2 u_0}{\partial t^2} = 0 \tag{15}$$

Eq. (15) can be simplified as

$$\frac{\partial}{\partial x}\left(\left(E(x)A\left(\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right)\times\left(1+\frac{\partial u_0}{\partial x}\right)\right) -\rho(x)A\frac{\partial^2 u_0}{\partial t^2}+\mu\frac{\partial^2}{\partial x^2}\left(\rho(x)A\frac{\partial^2 u_0}{\partial t^2}\right)=0 \quad (16)$$

Eq. (16) can be expanded and rewritten as bellow

$$\left(A(x)\frac{dE(x)}{dx}\right)\frac{\partial u_0}{\partial x}+\left(E(x)A(x)\right)\frac{\partial^2 u_0}{\partial x^2} +\left(\frac{3}{2}A(x)\frac{dE(x)}{dx}\right)\left(\frac{\partial u_0}{\partial x}\right)^2 +\left(3E(x)A(x)\right)\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} +\left(\frac{1}{2}A(x)\frac{dE(x)}{dx}\right)\left(\frac{\partial u_0}{\partial x}\right)^3 +\left(\frac{3}{2}E(x)A(x)\right)\left(\frac{\partial^2 u_0}{\partial x^2}\right)\left(\frac{\partial u_0}{\partial x}\right)^2 +\left(\mu A(x)\frac{d^2\rho(x)}{dx^2}-\rho(x)A(x)\right)\frac{\partial^2 u_0}{\partial t^2} +\left(\mu\rho(x)A(x)\right)\frac{\partial^4 u_0}{\partial x^2\partial t^2}=0 \quad (17)$$

It is worth to note that by considering Eqs. (10) and (14), the local and nonlocal boundary conditions become the same and are given by

$$\left(E(x)A\left(\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^2\right)\right)\left(1+\frac{\partial u_0}{\partial x}\right)\delta u_0\Big|_0^L=0 \quad (18)$$

$$\Rightarrow\left(\frac{\partial u_0}{\partial x}+\frac{3}{2}\left(\frac{\partial u_0}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u_0}{\partial x}\right)^3\right)\delta u_0\Big|_0^L=0$$

2.2. Free Vibration Analysis

2.2.1. Linear Free Vibration Analysis

Ignoring the nonlinear components in Eqs. (17) and (18) makes it possible to analyze the nonlocal linear free longitudinal vibration of AFG nano-size rods. The linear equations are as follows

$$\left(\frac{dE(x)}{dx}\right)\frac{\partial u_0}{\partial x}+\left(E(x)\right)\frac{\partial^2 u_0}{\partial x^2} +\left(\mu\frac{d^2\rho(x)}{dx^2}-\rho(x)\right)\frac{\partial^2 u_0}{\partial t^2} +\left(\mu\rho(x)\right)\frac{\partial^4 u_0}{\partial x^2\partial t^2}=0 \quad (19)$$

$$\left(\frac{\partial u_0}{\partial x}\right)\delta u_0\Big|_0^L=0 \quad (20)$$

Next, it is assumed that the AFG nano-size rod has a harmonic longitudinal displacement as

$$u_0(x,t)=U(x)e^{I\omega_L^{nl}t} \quad (21)$$

where $I=\sqrt{-1}$ is the imaginary unit, and ω_L^{nl} is the nonlocal natural axial frequency. By substituting Eq. (21) into Eqs. (19) and (20), the governing equation of motion and the boundary conditions in terms of the displacement are obtained as

$$\left(E(x)-\mu\rho(x)(\omega_L^{nl})^2\right)\frac{d^2U(x)}{dx^2} +\left(\frac{dE(x)}{dx}\right)\frac{dU(x)}{dx} +\left(\rho(x){\omega_L^{nl}}^2-\mu\frac{d^2\rho(x)}{dx^2}(\omega_L^{nl})^2\right)U(x)=0 \quad (22)$$

$$\frac{dU(x)}{dx}\delta U(x)\Big|_0^L=0 \Rightarrow \begin{cases} U(0)=U(L)=0 & Fi-Fi\ BC \\ U(0)=\dfrac{dU(L)}{dx}=0 & Fi-Fr\ BC \end{cases} \quad (23)$$

Since the coefficients of Eq. (22) are dependent on x, it is not possible to solve it analytically. To solve Eq. (22) and obtain the natural frequencies and corresponding mode shapes of AFG nanorods, the harmonic differential quadrature method (HDQM) [76], an accurate and useful method for analysis of structural components, is utilized in this study. Consider the domain of structure as $0<x<L$ which is discretized by N points along x-coordinate. If $F(x)$ represents the deformation function, then its derivative with respect to x at all points x_i can be expressed discretely as

$$\frac{d^jF(x_i)}{dx^j}=\sum_{k=1}^{N}A_{ik}^{(j)}F(x_k);\quad j=1,2,3,\ldots \quad (24)$$

in which $A_{ik}^{(j)}$ is the weighting coefficient in conjunction to the jth order derivative of $F(x)$, at the discrete point x_i.

Now, applying Eq. (24) to Eqs. (22)-(23) results in the discretized form of the governing equation and boundary conditions as follow

$$\left(E(x_i)-\mu\rho(x_i){\omega_L^{nl}}^2\right)\sum_{k=1}^{N}A_{ik}^{(2)}U(x_k) +\left(\sum_{k=1}^{N}A_{ik}^{(1)}E(x_k)\right)\sum_{k=1}^{N}A_{ik}^{(1)}U(x_k) +{\omega_L^{nl}}^2\left(\rho(x_i)-\mu\sum_{k=1}^{N}A_{ik}^{(2)}\rho(x_k)\right)U(x_i) =0 \quad (25)$$

$$\begin{cases} U(x_1)=U(x_N)=0; & Fi-Fi\ BC \\ U(x_1)=\sum_{k=1}^{N} A_{Nk}^{(1)}U(x_k)=0 & Fi-Fr\ BC \end{cases} \tag{26}$$

Arranging the discretized form of Eq. (25) in matrix form, considering the discretized forms of Eq. (26), and separating the domain ($\{V_D\}$) and boundary ($\{V_B\}$) degrees of freedom (DOF) lead to the following assembled matrix

$$\begin{bmatrix} [K_{BB}] & [K_{BD}] \\ [K_{DB}] & [K_{DD}] \end{bmatrix} \begin{Bmatrix} \{V_B\} \\ \{V_D\} \end{Bmatrix} = \omega_L^{nl^2} \begin{bmatrix} 0 & 0 \\ [M_{DB}] & [M_{DD}] \end{bmatrix} \begin{Bmatrix} \{V_B\} \\ \{V_D\} \end{Bmatrix} \tag{27}$$

where

$$\{V_B\} = \{U(x_1), U(x_N)\} \tag{28}$$

$$\{V_D\} = \{U(x_2), U(x_3), \dots, U(x_{N-2}), U(x_{N-1})\} \tag{29}$$

The nonlocal linear longitudinal frequencies, ω_L^{nl} and corresponding mode shapes, $U(x) = \{U(x_1),\ V_D,\ U(x_N)\}^T$ can be obtained from Eq. (27) after doing some mathematical simplification as

$$([M_{DD}]-[M_{DB}][K_{BB}]^{-1}[K_{BD}])^{-1}([K_{DD}] - [K_{DB}][K_{BB}]^{-1}[K_{BD}]) = \omega_L^{nl^2}\{V_D\} \tag{30}$$

2.2.2. Nonlinear Free Vibration Analysis Using the Method of Multiple Scales

For nonlinear free vibration analysis of nonlocal AFG nanorods, Galerkin's method is used to convert Eq. (17) to an ordinary differential equation. To this end, the longitudinal displacement of the n^{th} mode of the AFG nanorod can be considered as

$$u_0(x,t) = U(x)T(t) \tag{31}$$

where U(x) is the nonlocal normalized linear mode shape which can be obtained from Eqs. (27)-(30) and T(t) is a time dependent function to be determined. Substituting Eq. (31) into Eq. (17) results in the following ordinary differential equation

$$\begin{aligned} &\left(\left(\mu\frac{d^2\rho(x)}{dx^2}-\rho(x)\right)U(x)+\left(\mu\rho(x)\frac{d^2U}{dx^2}\right)\right)\ddot{T}(t)+\left(\frac{dE(x)}{dx}\frac{dU}{dx}+E(x)\frac{d^2U}{dx^2}\right)T(t) \\ &\quad+\left(\frac{3}{2}\frac{dE(x)}{dx}\left(\frac{dU}{dx}\right)^2+3E(x)\frac{dU}{dx}\frac{d^2U}{dx^2}\right)T(t)^2 \\ &\quad+\left(\frac{1}{2}\frac{dE(x)}{dx}\left(\frac{dU}{dx}\right)^3+\frac{3}{2}E(x)\frac{d^2U}{dx^2}\left(\frac{dU}{dx}\right)^2\right)T(t)^3=0 \end{aligned} \tag{32}$$

Then, multiplying Eq. (32) by $U(x)$ and integrating from x=0 to x=L result in

$$\ddot{T}(t)+\alpha_0T(t)+\alpha_1T(t)^2+\alpha_2T(t)^3=0 \tag{33}$$

where

$$\begin{aligned} \alpha_0 &= \frac{\int_0^L\left(\frac{dE(x)}{dx}\frac{dU}{dx}+E(x)\frac{d^2U}{dx^2}\right)U(x)\,dx}{\int_0^L\left(\left(\mu\frac{d^2\rho(x)}{dx^2}-\rho(x)\right)U(x)+\left(\mu\rho(x)\frac{d^2U}{dx^2}\right)\right)U(x)dx} \\ \alpha_1 &= \frac{\int_0^L\left(\frac{3}{2}\frac{dE(x)}{dx}\left(\frac{dU}{dx}\right)^2+3E(x)\frac{dU}{dx}\frac{d^2U}{dx^2}\right)U(x)dx}{\int_0^L\left(\left(\mu\frac{d^2\rho(x)}{dx^2}-\rho(x)\right)U(x)+\left(\mu\rho(x)\frac{d^2U}{dx^2}\right)\right)U(x)dx} \\ \alpha_2 &= \frac{\int_0^L\left(\frac{1}{2}\frac{dE(x)}{dx}\left(\frac{dU}{dx}\right)^3+\frac{3}{2}E(x)\frac{d^2U}{dx^2}\left(\frac{dU}{dx}\right)^2\right)U(x)dx}{\int_0^L\left(\left(\mu\frac{d^2\rho(x)}{dx^2}-\rho(x)\right)U(x)+\left(\mu\rho(x)\frac{d^2U}{dx^2}\right)\right)U(x)dx} \end{aligned} \tag{34}$$

Eq. (33) needs two initial conditions as follows

$$T(0) = T_{max} \tag{35}$$

$$\dot{T}(0) = 0 \tag{36}$$

where T_{max} is a maximum amplitude corresponding to the time dependent function T(t).

The method of multiple scales is implemented to solve Eq. (33). Before that, the following non-dimensional quantities are introduced

$$\bar{T} = \frac{T}{T_{max}} \qquad \bar{t} = \sqrt{\alpha_0}\, t \tag{37}$$

By substituting Eq. (37) into Eqs. (33), (35) and (36), the non-dimensional second order ordinary differential equation and initial conditions are obtained as

$$\ddot{\bar{T}}(t) + \bar{T}(t) + \bar{\alpha}_1 \bar{T}(t)^2 + \bar{\alpha}_2 \bar{T}(t)^3 = 0 \tag{38}$$

$$\bar{T}(0) = 1 \tag{39}$$

$$\dot{\bar{T}}(0) = 0 \tag{40}$$

in which $\bar{\alpha}_1 = (\alpha_1 T_{max})/\alpha_0$ and $\bar{\alpha}_2 = (\alpha_2 T_{max}^2)/\alpha_0$.

The method of multiple scales considering a second-order uniform expansion of the solution of Eq. (38) is desired as

$$\bar{T}(t) = \sum_{n=1}^{3} \varepsilon^n \bar{T}_n(q_0, q_1, q_2) + \cdots \tag{41}$$

where ε is a small parameter measuring the amplitude of oscillation. In addition, we consider a fast time scale, $q_0 = t$, denoting the main oscillatory behavior, and slow time scales $q_n = \varepsilon^n t, n \geq 1$, implying the amplitude and phase modulation.

Since the form of Eq. (38) is the same as the one obtained for nonlinear free transverse vibration of nanobeams [20], the procedure of extracting the nonlocal natural frequency equation of AFG nanorods is not presented here. If we follow the procedure presented in Ref. [20], the nonlocal natural frequency equation of AFG nanorods can be given as

$$\omega_{nl}^{nl} = \sqrt{\alpha_0}\left(1 + \frac{3}{8}\left(\bar{\alpha}_2 - \frac{10}{9}\bar{\alpha}_1^2\right)\right) \tag{42}$$

where ω_{nl}^{nl} is the nonlocal nonlinear natural frequency of the AFG nanorod, and the subscript and superscript *nl* denote the nonlinear and nonlocal, respectively.

3. Results and Discussion

Before presentation of results, it is worth to mention that for obtaining the nonlinear natural frequencies of AFG nanorods (Eq. (42)), the coefficients defined in Eq. (43) should be calculated. One of the parameters used in the coefficients is the linear normalized mode shapes (Eqs. (28) and (29)) which are obtained in discretized form. Therefore, we do a curve-fitting to obtain a function for each linear mode shape. This makes it possible to do mathematical operations on the functions to obtain nonlinear natural frequencies. In Fig. 3, linear mode shapes of AFG nano-size rod for various values of the gradient index are plotted for fixed-fixed (fi-fi) and fixed-free (fi-fr) end conditions.

3.1. Comparison Study

In this section, a comparison study is conducted to check the reliability of the present formulation and the analytical solution, and some numerical results are provided for the nonlinear vibration properties of AFG nano-size rods based on the nonlocal elasticity theory.

Table 1 gives a comparison between the natural frequencies of a fi-fi and a fi-fr boundary conditions presented by this work with those obtained by Nazemnezhad and Zare [70] for a homogeneous nanoscale rod. The both researches have used the method of multiple scale to obtain the natural frequencies. In the present study, an AFG nanorod is investigated while in Ref [70] a homogeneous one is desired. Table 1 shows a good agreement in results.

3.2. Benchmark Results

The benchmark results presented here are obtained for nanorods with the mechanical properties given in Table 2.

We first examine the variations of linear and nonlinear nonlocal frequencies versus AFG nanorod diameters. In this regard, the length of the AFG nanorod is equal to 10 (nm) and the corresponding diameter changes are equal to 1-2 (nm) with a step of 0.25. Since the change in diameter is directly related to the thickness of the AFG nanorod, we have tried to select its values so that the aspect ratio of the AFG nanorod is greater than 10, because in the calculation of nonlinear nonlocal frequencies, the Simple axial beam model has been used, which is suitable for this condition. The results show that with increasing AFG nanorod diameter, the linear nonlocal frequencies are constant, but the nonlinear nonlocal frequencies have increased, although this increase is not significant. Due to the fact that the AFG nanorod material leads to softness, with increasing the gradient index, linear and nonlinear nonlocal frequencies have decreased. Also, due to the fact that the system under study has a hardening effect, with increasing nonlinear amplitude, nonlinear nonlocal frequencies have increased. The results of these changes are presented in Tables 3 and 4, respectively, for fi-fi and fi-fr boundary conditions in terms of the first frequency mode of the AFG nanorod. As a results show, the general trend of changes is the same for both boundary conditions, but with the difference that in the fi-fi boundary conditions in the first mode, the effect of hardening is maintained, while for the fi-fr boundary conditions, we see a softening effect in the first mode.

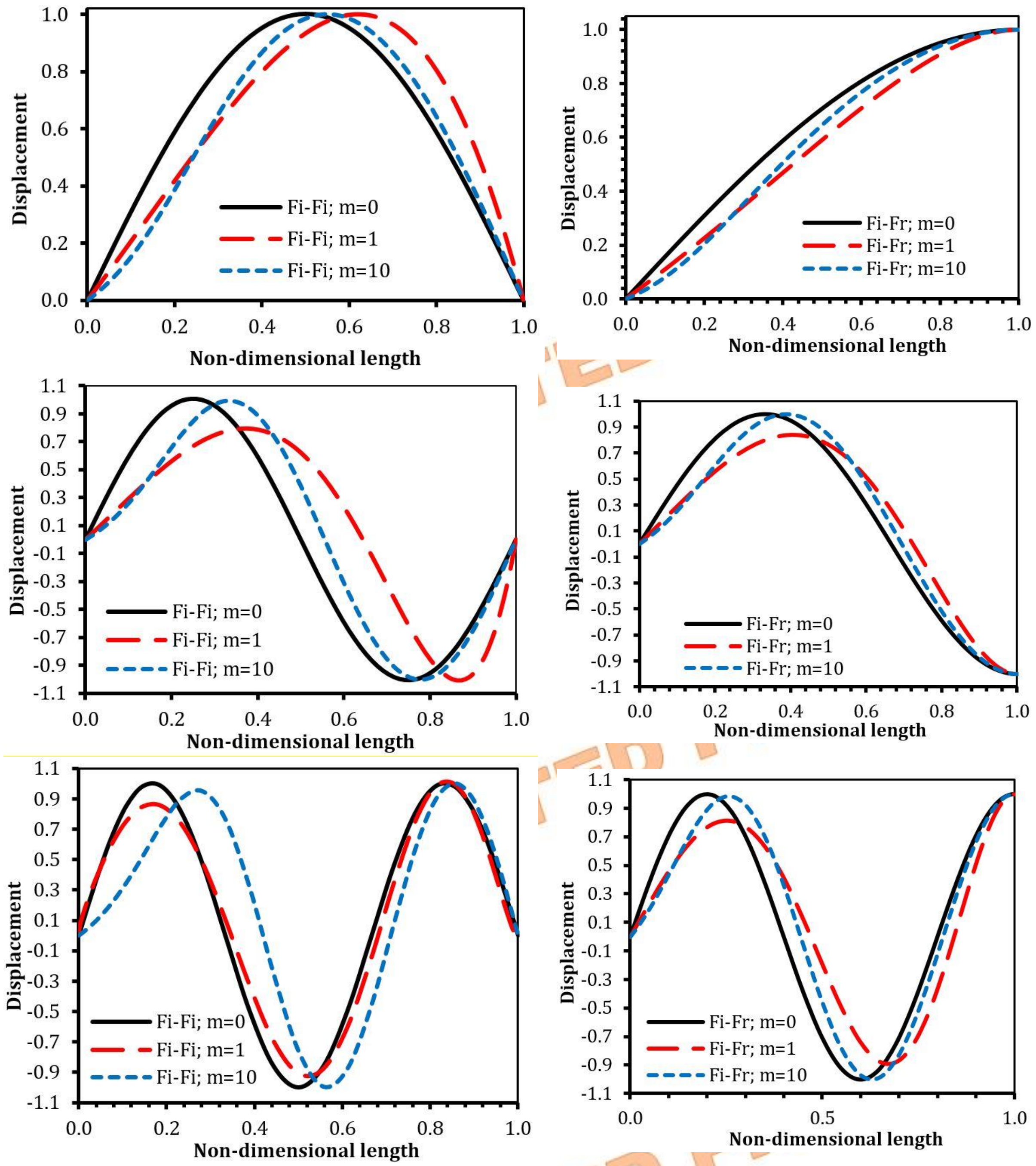

Fig. 3. The first three mode shapes of fixed-fixed and fixed-free nanorods for various values of the gradient index.

Table 1. Comparison study of the present work with Ref. [70] in terms of the first third natural frequencies (GHz) for AFG nanorod (μ=2 nm, R=2 nm, T_{max} = 2, m=0)

Length (nm)	Mode number	fi-fr BC (E_R/E_L = 1)		fi-fi BC (E_R/E_L = 1)	
		Present work	Ref. [70]	Present work	Ref. [70]
10	1	217.602	217.602	453.996	453.996
	2	621.698	621.698	859.916	859.916
	3	1007.970	1007.970	1270.760	1270.760
20	1	115.402	115.402	232.917	232.917
	2	339.767	339.767	453.996	453.996
	3	550.372	550.372	660.864	660.864

Table 2. Mechanical properties of AFG nanorod

Parameter	E_R (GPa)	E_L (GPa)	ρ_R (kg/m^3)	ρ_L (kg/m^3)	ν
value	70	210	2700	2370	0.3

Table 3. Variations of longitudinal frequencies versus diameters of AFG nanorod in terms of Fi-fi BCs, by considering L = 20 nm, n = 1

D	m = 0			m = 1			m = 10		
	μ = 0	μ = 1	μ = 2	μ = 0	μ = 1	μ = 2	μ = 0	μ = 1	μ = 2
Nonlinear amplitude (T_{max} = 0)									
1	234.806	231.974	229.242	178.002	175.633	173.344	139.397	137.577	135.823
1.25	234.806	231.974	229.242	178.002	175.633	173.344	139.397	137.577	135.823
1.5	234.806	231.974	229.242	178.002	175.633	173.344	139.397	137.577	135.823
1.75	234.806	231.974	229.242	178.002	175.633	173.344	139.397	137.577	135.823
2	234.806	231.974	229.242	178.002	175.633	173.344	139.397	137.577	135.823
Nonlinear amplitude (T_{max} = 1)									
1	235.010	232.176	229.441	178.161	175.787	173.494	139.503	137.681	135.926
1.25	235.125	232.289	229.553	178.250	175.874	173.578	139.562	137.739	135.983
1.5	235.265	232.427	229.690	178.359	175.980	173.681	139.635	137.810	136.053
1.75	235.430	232.591	229.852	178.488	176.106	173.803	139.720	137.895	136.136
2	235.621	232.780	230.038	178.637	176.251	173.943	139.819	137.992	136.232
Nonlinear amplitude (T_{max} = 2)									
1	235.621	232.780	230.038	178.637	176.251	173.943	139.819	137.992	136.232
1.25	236.080	233.233	230.486	178.995	176.598	174.280	140.057	138.226	136.462
1.5	236.641	233.787	231.033	179.432	177.023	174.692	140.347	138.512	135.926
1.75	237.303	234.441	231.680	179.948	177.525	175.179	140.691	138.849	136.743
2	238.068	235.196	232.427	180.544	178.104	175.741	141.087	139.239	137.458

Table 4. Variations of longitudinal frequencies versus diameters of AFG nanorod in terms of Fi-fr BCs, by considering L = 20 nm, n = 1

D	m = 0			m = 1			m = 10		
	μ = 0	μ = 1	μ = 2	μ = 0	μ = 1	μ = 2	μ = 0	μ = 1	μ = 2
Nonlinear amplitude (T_{max} = 0)									
1	117.627	117.266	116.908	98.468	98.145	97.825	70.243	70.008	69.7747
1.25	117.627	117.266	116.908	98.468	98.145	97.825	70.243	70.008	69.7747
1.5	117.627	117.266	116.908	98.468	98.145	97.825	70.243	70.008	69.7747
1.75	117.627	117.266	116.908	98.468	98.145	97.825	70.243	70.008	69.7747
2	117.627	117.266	116.908	98.468	98.145	97.825	70.243	70.008	69.7747
Nonlinear amplitude (T_{max} = 1)									
1	117.530	117.169	116.812	98.408	98.085	97.765	70.191	69.956	69.723
1.25	117.475	117.115	116.758	98.374	98.051	97.731	70.162	69.927	69.694
1.5	117.408	117.048	116.691	98.332	98.010	97.690	70.126	69.891	69.658
1.75	117.330	116.970	116.613	98.283	97.961	97.642	70.084	69.849	69.616
2	117.239	116.879	116.523	98.227	97.905	97.586	70.035	69.801	69.568
Nonlinear amplitude (T_{max} = 2)									
1	117.239	116.879	116.523	98.227	97.905	97.586	70.035	69.801	69.568
1.25	117.021	116.662	116.306	98.091	97.769	97.451	69.919	69.684	69.452
1.5	116.754	116.396	116.041	97.924	97.604	97.287	69.776	69.542	69.310
1.75	116.439	116.081	115.727	97.728	97.405	97.092	69.607	69.374	69.143
2	116.075	115.719	115.366	97.501	97.183	96.868	69.413	69.180	68.949

Next, the linear and nonlinear frequency changes of the AFG nanorod are studied based on the variations of the non-local parameter and the gradient index for the first and second AFG nanorod modes in terms of both fi-fi and fi-fr boundary conditions. These results are derived for AFG nanorods with a length of 10 (nm) and a diameter of 1 (nm) in terms of three nonlinear amplitudes of 0, 1 and 2. The results show that the decreasing effects of nonlocal parameter on linear and nonlinear frequencies for the first mode are negligible. Also, the decreasing slope of the graphs for the lower gradient index is higher, which indicates that the stricter the AFG nanorod, the greater the decreasing effects of the nonlocal parameter on its linear and nonlinear frequencies. Here, too, the trend of general results changes for both fi-fi and fi-fr boundary conditions is similar, except that the frequency-reducing effects of the non-local parameter for fi-fr boundary conditions are less than fi-fi. The detailed results of the study of these changes can be clearly seen in Figs 4a to 4f.

In this section, the longitudinal frequency changes of AFG nanorods based on nonlinear amplitudes for different non-local parameter values and gradient index are evaluated. In this evaluation, the length and diameter of the AFG nanorod are considered equal to 10 and 1 (nm), respectively. As shown in Fig 5, the variations of the longitudinal frequencies of the AFG nanorod versus the nonlinear amplitude changes are greater for the fi-fi than fi-fr boundary conditions. Also, the incremental effect of nonlinear nonlocal frequencies due to the increase of nonlinear amplitude for the second mode of AFG nanorod frequency is higher than the first mode, but this incremental effect will be more in higher modes (third mode onwards). In addition, the results show that the incremental effects of increasing the nonlinear amplitude are the same for different gradient index nanorod. In other words, the nonlinear effects of material changes along the AFG nanorod are neutral. The results of these studies are clearly shown in the various sections of Fig 5a, 5b, 5c and 5d for both boundary conditions.

Finally, the frequency changes of the AFG nanorod versus its length for different non-local parameter values and gradient index are presented in Fig 6. These results are derived for both fi-fi (Figs. 6a, 6b and 6c) and fi-fr (Figs. 6d, 6e and 6f) boundary conditions for an AFG nanorod with nonlinear amplitudes of 0, 1, and 2.

Figs 6a, 6b and 6c show that the decreasing effects of linear and nonlinear frequencies due to the non-local parameter in the fi-fi boundary conditions are greater than the fi-fr, which it has been observed in previous results. Another important point is that with increasing AFG nanorod length, linear and nonlinear nonlocal frequencies coincide, which shows that the effects of changes in material, nonlocal parameter and nonlinear amplitude on long AFG nanorods on its frequencies are negligible. The bottom line is that the larger the AFG nanorod length, the higher the nonlinear and nonlocal coefficients must be for the results to be significant. Also on thick nanorods, the effects of nonlinear factor are less than on thin nanorods, see Fig. 6.

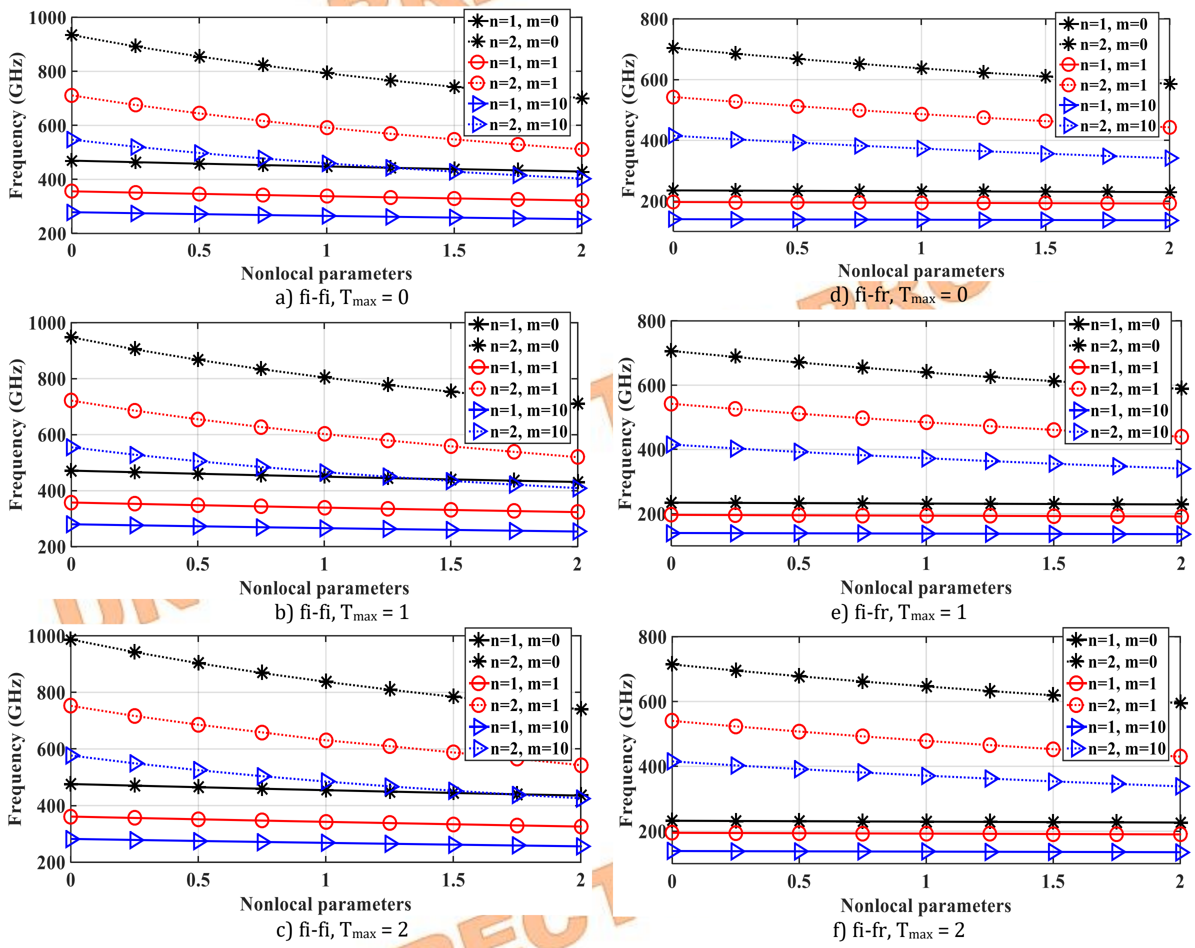

Fig. 4. Variations of longitudinal frequencies versus nonlocal parameters for various gradient indexes and mode numbers of AFG nanorod by considering L = 10 nm, D = 1 nm

4. Conclusions

In the present study, large amplitude free longitudinal vibration behaviour of axially functionally graded nanorod with material gradation is investigated. Different boundary conditions are considered namely fi-fi and fi-fr. However the present methodology can be employed for other type of classical and non-classical boundary as well. Hamilton's principle is utilized for the mathematical formulation and the boundary conditions, problem can be solved in two part, linear and nonlinear with the help of Harmonic Differential Quadrature and Multiple Scale methods, respectively. The methodology is general in nature as it can be applied for other type of material gradation and rod pattern. The obtained results are confirmed from previously published results and were found to be in good agreement. Results pertaining to various AFG nanorod parameter for fi-fi and fi-fr boundary conditions are furnished as highlight curves for the fundamental modes. For all combinations of the system parameters hardening type of nonlinearity is observed in terms of fi-fi boundary conditions while, the softening type is demonstrated for initial modes in terms of fi-fr boundary conditions. Nonlinear nonlocal analysis of rod-type structural components of NEMS (Nano-Electro-Mechanical-Systems) is a topic of major interest in Engineering and Medical sciences.

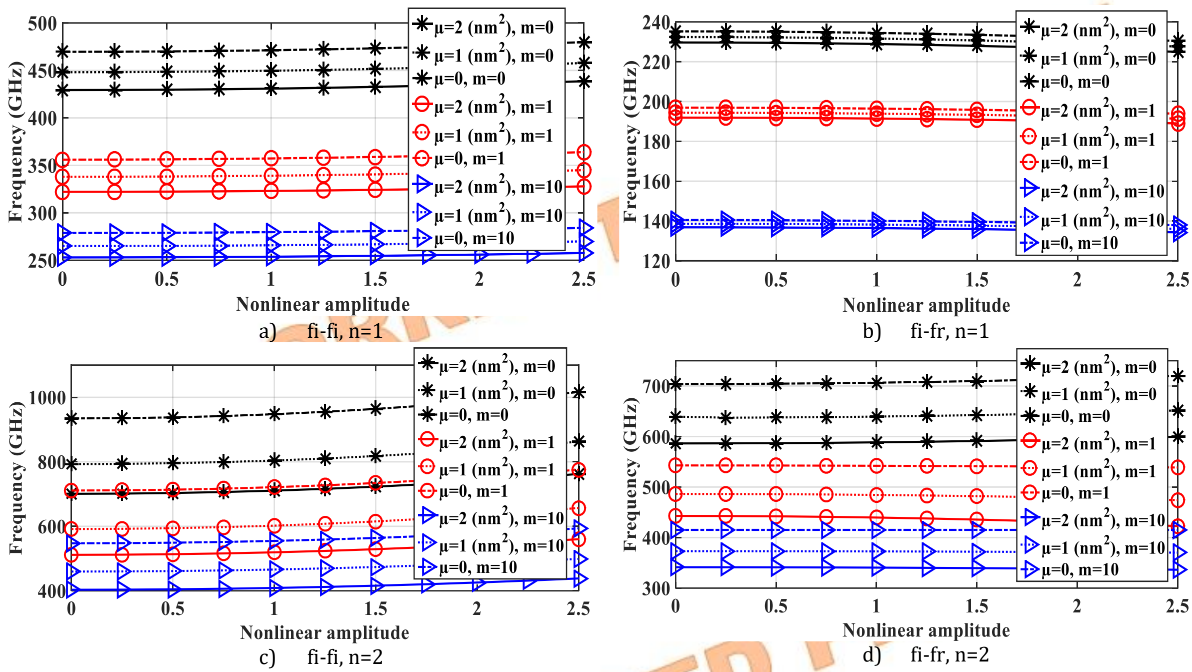

Fig. 5. Variations of longitudinal frequencies versus nonlinear amplitudes for various nonlocal parameters and gradient indexes of AFG nanorod by considering L = 10 nm, D = 1 nm

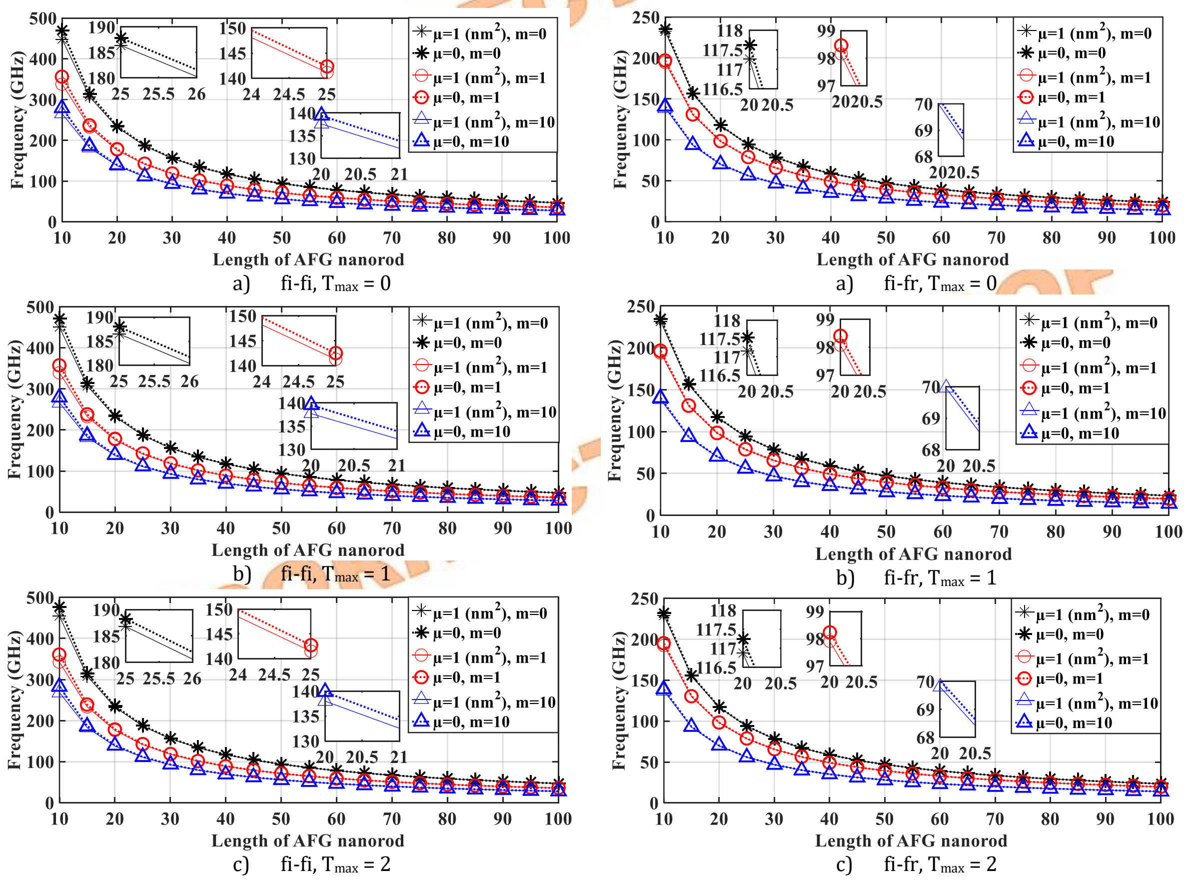

Fig. 6. Variations of longitudinal frequencies versus length of AFG nanorod for various nonlocal parameters and gradient indexes

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